Euclidean distance between flats in geometric algebra.v2

A. Cortzen, ac.ga.ca by gmail.com

Abstract

This is a simple calculation of a shortest linear segment connecting two disjuncts flats.
The segment is proven orthogonal to both flats.
Possible solutions are considered.
The formula has no exceptions and is proved in The homogeneous Model and The conformal Model.
The symbolic is the same as in [1], and the reader is assumed familiar with some of the essential results from this book.

The homogeneous Model

Lemma Let $X_i = (e_0 + \mathbf{p}_i) \wedge \mathbf{A}_i$, $i = 1, 2$ *be flats,* $\mathbf{J} = \mathbf{A}_1 \bigcup \mathbf{A}_2$ *and* $\mathbf{d} = (\mathbf{J} \wedge (\mathbf{p}_2 - \mathbf{p}_1))/\mathbf{J}$. *Then*

1. $d = \|\mathbf{d}\| = \|\mathbf{J} \wedge (\mathbf{p}_2 - \mathbf{p}_1)\| / \|\mathbf{J}\|$ *is the shortest distance between* X_1 *and* X_2
2. $\mathbf{d}$ *is orthogonal to* X_1 *and* X_2
3. *Translation by* $\mathbf{d}$ *takes some point* q_1 *in* X_1 *into a point* q_2 *in* X_2.
   The possible points q_1 *are the points in the flat* $X_1 \bigcap \mathrm{T}_{-\mathbf{d}}[X_2]$ *with direction* $\mathbf{M} = \mathbf{A}_1 \bigcap \mathbf{A}_2$.

Proof: (You may as an example visualize two non-intersecting lines in 3D)
The well-known relations for the meet $\mathbf{M}$ and the join $\mathbf{J}$ are
$\mathbf{A}_1 = \mathbf{B}_1 \wedge \mathbf{M}$, $\mathbf{A}_2 = \mathbf{M} \wedge \mathbf{B}_2$ and $\mathbf{J} = \mathbf{B}_1 \wedge \mathbf{M} \wedge \mathbf{B}_2$, if $\mathbf{B}_1 \wedge \mathbf{B}_2 \neq 0$
For the rejection $\mathbf{d}$ of $\mathbf{p}_2 - \mathbf{p}_1$ by $\mathbf{J}$ we can select vectors $\mathbf{b}_i \subseteq \mathbf{B}_i$ and $\mathbf{m} \subseteq \mathbf{M}$, such that $\mathbf{d} = \mathbf{p}_2 - \mathbf{p}_1 + \mathbf{b}_1 + \mathbf{m} + \mathbf{b}_2$.
This equation is changed to $\mathbf{p}_2 - \mathbf{p}_1 = \mathbf{d}$ by the replacements $\mathbf{p}_1 \to \mathbf{p}_1 + \mathbf{b}_1 + \mathbf{m}$ and $\mathbf{p}_2 \to \mathbf{p}_2 - \mathbf{b}_2$. Translating $p_1 = e_0 + \mathbf{p}_1$ by $\mathbf{d}$ now results in the point $p_2 = e_0 + \mathbf{p}_2$.
For points $a_i = e_0 + \mathbf{p}_i + \mathbf{a}_i \subseteq X_i$, i.e. $\mathbf{a}_i \subseteq \mathbf{A}_i$, we finally have $(a_2 - a_1)^2 = (\mathbf{d} + (\mathbf{a}_2 - \mathbf{a}_1))^2 = \mathbf{d}^2 + (\mathbf{a}_2 - \mathbf{a}_1)^2 \geq \mathbf{d}^2$, as $(\mathbf{a}_2 - \mathbf{a}_1) \cdot \mathbf{d} = 0$.

Remarks:
We may also start from dual flats, where $\mathbf{C}_i = \mathbf{A}_i^\star$ is known, and use $\mathbf{J}^\star \equiv \mathbf{C}_1 \bigcap \mathbf{C}_2$.
If dual flats are given, we may use $\mathbf{J}^\star = \mathbf{C}_1 \bigcap \mathbf{C}_2$ with known $\mathbf{C}_i = \mathbf{A}_i^\star$.
Of course simplifications follows in special cases, e.g.

4. *If* $\mathbf{A}_1 \wedge \mathbf{A}_2 \neq 0$, *then* $\mathbf{J} = \mathbf{A}_1 \wedge \mathbf{A}_2$, *and* $\mathbf{d} = (\mathbf{A}_1 \wedge \mathbf{A}_2 \wedge (\mathbf{p}_2 - \mathbf{p}_1))/(\mathbf{A}_1 \wedge \mathbf{A}_2)$.
5. *If* $\mathbf{A}_1 \equiv \mathbf{A}_2$, *then* $\mathbf{J} = \mathbf{A}_1$, *and* $\mathbf{d} = (\mathbf{A}_1 \wedge (\mathbf{p}_2 - \mathbf{p}_1))/\mathbf{A}_1$.

The conformal Model

Lemma Let $X_i = (o + \mathbf{p}_i) \wedge \mathbf{A}_i \wedge \infty$, $i = 1, 2$ *be flats,* $\mathbf{J} = \mathbf{A}_1 \bigcup \mathbf{A}_2$ *and* $\mathbf{d} = (\mathbf{J} \wedge (\mathbf{p}_2 - \mathbf{p}_1))/\mathbf{J}$. *Then*

1. $d = \|\mathbf{d}\| = \|\mathbf{J} \wedge (\mathbf{p}_2 - \mathbf{p}_1)\| / \|\mathbf{J}\|$ *is the shortest distance between* X_1 *and* X_2
2. $\mathbf{d}$ *is orthogonal to* X_1 *and* X_2
3. *Translation by* $\mathbf{d}$ *takes some point in* X_1 *into a point in* X_2.
   The possible points q_1 *are the points in the flat* $X_1 \bigcap \mathrm{T}_{-\mathbf{d}}[X_2]$, *which has direction* $\mathbf{M} = \mathbf{A}_1 \bigcap \mathbf{A}_2$.

The proof is similar to the previous and for completeness given in full form :
The well-known relations for the meet $\mathbf{M} = \mathbf{A}_1 \bigcap \mathbf{A}_2$ and the join $\mathbf{J}$ are
$\mathbf{A}_1 = \mathbf{B}_1 \wedge \mathbf{M}$, $\mathbf{A}_2 = \mathbf{M} \wedge \mathbf{B}_2$, and $\mathbf{J} = \mathbf{B}_1 \wedge \mathbf{M} \wedge \mathbf{B}_2$.
For the rejection $\mathbf{d}$ of $\mathbf{p}_2 - \mathbf{p}_1$ by $\mathbf{J}$ we can select vectors $\mathbf{b}_i \subseteq \mathbf{B}_i$ and $\mathbf{m} \subseteq \mathbf{M}$, such that $\mathbf{d} = \mathbf{p}_2 - \mathbf{p}_1 + \mathbf{b}_1 + \mathbf{m} + \mathbf{b}_2$.
This equation is changed to $\mathbf{p}_2 - \mathbf{p}_1 = \mathbf{d}$ by the replacements $\mathbf{p}_1 \to \mathbf{p}_1 + \mathbf{b}_1 + \mathbf{m}$ and $\mathbf{p}_2 \to \mathbf{p}_2 - \mathbf{b}_2$, which fixes the flats and $\mathbf{d}$. Furthermore translating $\mathrm{T}_{\mathbf{p}_1}[o]$ by $\mathbf{d}$ now results in the point $\mathrm{T}_{\mathbf{p}_2}[o]$.
For points $a_i = \mathrm{T}_{\mathbf{p}_1}[o] \subseteq X_i$, i.e. $\mathbf{a}_i \subseteq \mathbf{A}_i$, we finally have $(a_2 - a_1)^2 = (\mathbf{d} + (\mathbf{a}_2 - \mathbf{a}_1))^2 = \mathbf{d}^2 + (\mathbf{a}_2 - \mathbf{a}_1)^2 \geq \mathbf{d}^2$.

References

[1] L. Dorst, D. Fontijne, and S. Mann. Geometric Algebra for Computer Science,
An Object-Oriented Approach to Geometry. Morgan Kaufman, 2007.