Direct Construction of Clifford and Geometric Algebras and their basic algebraic Structures v.2.0.

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Abstract

This is a simple way rigorously to construct Grassmann, Clifford and Geometric Algebras, allowing degenerate bilinear forms, infinite dimension, using fields or modules (characteristic 2 with limitations for certain Clifford algebras), and characterize the algebras in a coordinate free form.
The construction is done in an orthogonal basis, and the algebras characterized by universality.
Most properties are with short proofs provides a clear foundation for application of the algebras.
A comprehensive formula collection is established.
Various conditions for non-universality are established. For such algebras conditions for reversion and Clifford conjugation are found.
Some properties or proofs might be new in this context, e.g. factor expansion and parallel projection.
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Chapter 1 Definition of Clifford algebras

Introduction

Often proof of the existence of Grassmann or Clifford algebras are bypassed, or Chevalley's tensor approach is taken. Pure mathematical books may present a lot of structures before coming to these algebras [2,3,5]. Here a direct approach is described.

Preliminaries

Our starting-point may e.g. be the real number field $\mathbb{R}$, a linear space $V = \mathbb{R}^n$ with basis $e_1 = (1, 0, \ldots 0)$, $e_2 = (0, 1, \ldots 0) \ldots$, and indices $M = \{1, 2, \ldots, n\}$ usually ordered by <. Also used is a quadratic mapping $q(i) = B(e_i, e_i)$, where B is a bilinear form on V with diagonal form in the basis $(e_i \mid i \in M)$.

The basic idea behind Grassmann and Clifford algebras is, that products may give new elements, e.g. $e_1\, e_2 = e_{\{1,2\}}$. Here a common construction will cover both algebras.

In $\mathbb{R}^n$ the new product should fulfill generator equations $e_1\, e_2 = -e_2\, e_1$, $e_i\, e_i = q(e_i) \in \mathbb{R}$ and be associative. Then a product may be reordered and reduced to get a standard form without repetitions, as in

$e_{\{1,2\}}\, e_3\, e_1\, e_2 = -e_1\, e_2\, e_1\, e_3\, e_2 = e_1\, e_1\, e_2\, e_3\, e_2 = -e_1\, e_1\, e_2\, e_2\, e_3 = -q(e_1)\, q(e_2)\, e_3$

The product properties gives a dimension $\leq 2^n$, as there are 2^n subsets of M.

The first goal of the algebra construction is to equip $W = \mathbb{R}^{2^n}$ with a Clifford product.

A basis for W is $(e_K \mid K \subseteq M)$. Submodules of W are the scalars $R\, e_\emptyset$ and V by identifying e_i with $e_{\{i\}}$.

Sets as indices gives a compact construction. It can indeed be used together with multiindex: $e_4\, e_3\, e_5 = e_{(4,3,5)} = -e_{\{4,3,5\}} = -e_{\{3,4,5\}}$.

Also allowed $e_{43} = -e_{34} = -e_{\{3,4\}}$, when misunderstanding is not probable.

Conventions

The following notation and definitions will be used. An algebra A is a linear space equipped with a bilinear and associative composition having a unit 1_A. An algebra morphism is supposed to map unit to unit. All the algebras are over the same set of scalars, R. Silently x, y will be elements in a linear space V and X, Y elements in the algebra at hand.

The cardinality of the set K is denoted $|K|$. Multiindices are always subsets taken from a totally ordered set, and as such totally ordered.

For a index set H we use $k < H$ in the meaning $\forall\, h \in H (k < h)$, implying $k < \emptyset$ is true.

A product for increasing indices is marked with $\uparrow$ and decreasing with $\downarrow$.

Moreover $\uparrow$ and $\in$ may be omitted like in $\Pi_{i \in I, \uparrow}\, a_i = \Pi_I\, a_i$.

$H \triangle J = (H \bigcup J) \setminus (H \bigcap J)$ is the symmetric set difference, which is associative.

Definition of a Clifford algebra

To make the exposition general we may assume

1. $\mathbb{K}$ is a commutative ring of *scalars* with unit $1 \neq 0$, and $\mathbb{K}^\times = \mathbb{K}\backslash\{0\}$. Small Greek characters have usually scalar values.
2. V is a free unitary left $\mathbb{K}$-module
3. $B : V \times V \to \mathbb{K}$ is a bilinear form with diagonal form in the basis $(e_i \mid i \in M)$, $q(e_i) = B(e_i, e_i)$ and M is totally ordered by a relation $<$.
4. $W = \oplus_{\mathcal{F}} \mathbb{K}$, where $\mathcal{F}$ is the set of finite subsets of index set M.

The standard basis for W is $(e_K \mid K \in \mathcal{F})$. By identifying e_i with $e_{\{i\}}$ we consider V a submodule of W.

The scalars $\mathbb{K}\, e_\emptyset$ are identified with $\mathbb{K}$ where it does not give problems.

NB: $V = \{0\} \Rightarrow W = \mathbb{K}\, e_\emptyset = \mathbb{K}$

NB: Instead of W any other free unitary $\mathbb{K}$-module with $\left(\Pi_K\, e_k \mid K \in \mathcal{F}\right)$ as basis can be used.

NB: If $\mathbb{K}$ is a field with characteristic different from 2, and V has finite dimension, then any symmetric bilinear form on V has an orthogonal basis.

Definition 1.1. A Clifford algebra U over B, is an algebra containing V, such that

1. $\forall_{x \in V} : x^2 = B(x, x)\, 1_U$

2. V generates U

3. $V \bigcap \mathbb{K}\, 1_U = \{0\}$.

Let $\mathcal{A}(V, B)$ *be the category of Clifford algebras over B.*

U is called (initial-) ***universal*** *in* $\mathcal{A}(V, B)$*, if*

4. Any linear mapping $f : V \to A$ *into an algebra A, such that* $f(x)^2 = B(x, x)\, 1_A$*, has a unique extension to algebra morphism* $F : U \to A$*. This extension is called the universal extension.*

Theorem 1.2. Assume algebras U_i *are universal in* $\mathcal{A}(V_i, B_i)$*,* $f : V_1 \to V_2$ *is* $\mathbb{K}$*-linear and* $f : V_1 \to V_2$ *is* $\mathbb{K}$*-linear.*

Then f has a unique extension, $F : U_1 \to U_2$ *to an algebra morphism, which is an isomorphism, if f is bijective.*

Proof: Universality gives a unique extension of $f : V_1 \to U_2$ to an algebra morphisms $F : U_1 \to U_2$.
If f is bijective, then from f^{-1} we likewise get $G : U_2 \to U_1$.
As $F \circ G$ is identity on V_2, uniqueness of extension by universality implies $\mathrm{id}_{U_2} = F \circ G$, and likewise $\mathrm{id}_{U_1} = G \circ F$. Thus F and G are isomorphisms.

Now the following two corollaries are obvious.

Corollary 1.3. An universal algebra in $\mathcal{A}(V, B)$ is uniquely determined aside from isomorphisms fixing V.

Corollary 1.4. Assume algebras U_i are universal in $\mathcal{A}(V_i, B_i)$ and $f_i : V_i \to V_{i+1}$ is $\mathbb{K}$-linear.
Let $F_i : U_i \to U_{i+1}$ be the unique extensions to algebra morphisms.
Then the unique extension of $f_k \circ \ldots \circ f_1$ to an algebra morphism is $F_k \circ \ldots \circ F_1$.

Chapter 2 Construction of a universal Clifford algebra

Construction

The construction needs help functions α, β. If H, J are index sets, then ordering the list concatenation of H and J ascending by swapping neighbors contributes a factor to α of -1 each time. It can be done successively starting in the highest end of H comparing for swapping from the lowest end of J. This gives the factor $\alpha(H, J)$. Then identical neighbors is multiplied and removed, which gives a factor $q(e_i)$ absorbed in β.

Definition 2.1. Let for sets $H, J \in \mathcal{F}$

$$\alpha(H, J) = \Pi\,(-1) \text{ for } (h, j) \in H \times J \text{ and } h > j$$

$$\beta(H, J) = \Pi\, q(e_i) \textit{ for } i \in H \cap J\ ,$$

$$\sigma = \alpha\, \beta$$

NB: Allowed here is to use a single element i in M, instead of the set $\{i\}$.

Proof of existence of a Clifford algebra over *B*.

Theorem 2.2. Define a product $(X, Y) \to X\,Y$ in W by $e_H\, e_J = \sigma(H, J)\, e_{H \triangle J}$ and bilinearity.
Then W becomes a Clifford algebra in $\mathcal{A}(V, B)$.

Proof: In definition 1.1 properties 2-3 are true by construction. Now to property 1.
As the associative law $(x\,y)\,z = x\,(y\,z)$ is multilinear, it needs only be verified for basis elements:

$$(e_H\, e_J)\, e_K = \sigma(H, J)\, e_{H \triangle J}\, e_K = \sigma(H, J)\, \sigma(H \triangle J, K)\, e_{(H \triangle J) \triangle K}\ ,$$

$$e_H\, (e_J\, e_K) = \sigma(J, K)\, e_H\, e_{J \triangle K} = \sigma(H, J \triangle K)\, \sigma(J, K)\, e_{H \triangle (J \triangle K)}$$

Equality of the two expression follow from Δ being associative, and the lemma below.
As $e_\emptyset\, e_K = \sigma\big(\emptyset, K\big)\, e_K = e_K = \sigma\big(K, \emptyset\big)\, e_K = e_K\, e_\emptyset$, we get $e_\emptyset = 1_W = 1$.
Moreover and $e_i\, e_i = \sigma(i, i)\, e_\emptyset = q(e_i)\, e_\emptyset$ and $i < j \Rightarrow (e_i\, e_j = \sigma(i, j)\, e_{\{i,j\}} = e_{\{i,j\}}$ and $e_j\, e_i = -e_{\{i,j\}} = -e_i\, e_j)$.
Hence $x = \Sigma_i\ \lambda_i\, e_i$ imply $x^2 = \Sigma_{i,j}\ \lambda_i\, \lambda_j\, e_i\, e_j = \Sigma\, \lambda_i^2\, e_i^2$ and finally $B(x, x) = \Sigma_{i,j}\ \lambda_i\, \lambda_j\, B(e_i, e_j) = \Sigma\, \lambda_i^2\, B(e_i, e_i) = \Sigma\, \lambda_i^2\, e_i^2 = x^2$.

Lemma 2.3. $\sigma(H, J)\, \sigma(H \triangle J, K) = \sigma(H, J \triangle K)\, \sigma(J, K)$,

Proof: As $\alpha^2 = 1$ we get $\alpha(H \triangle J, K) = \alpha(H \backslash J, K)\, \alpha(J \backslash H, K)\, \alpha(J \cap H, K)^2 = \alpha(H, K)\, \alpha(J, K)$ and likewise $\alpha(H, J \triangle K) = \alpha(H, J)\, \alpha(H, K)$. Thus the equation is obvious for the α-part of σ.

Using the temporary notation like $\overline{K} = M \backslash K$ and $[HJK]$ for the product of $q(e_i)$ for $i \in H \cap J \cap K$, then

$\beta(H, J) = [HJK]\big[HJ\overline{K}\big]$, and

$\beta(H, J)\, \beta(H \triangle J, K) = [HJK]\big[HJ\overline{K}\big]\big[H\overline{J}K\big]\big[\overline{H}JK\big] = \beta(H, J \triangle K)\, \beta(J, K)$.

Corollary 2.4. $e_I = \Pi_I\, e_i$.

Proof: Use induction and $H < j \Rightarrow e_H\, e_j = \sigma(H, \{j\})\, e_{H \cup \{j\}} = e_{H \cup \{j\}}$.

Proof of existence of a *universal* Clifford algebra over *B*.

Theorem 2.5. 1. W is universal in $\mathcal{A}(V, B)$.
Hence W is uniquely determined by universality in $\mathcal{A}(V, B)$ *aside from isomorphism.*
W is denoted $C\ell(B, V)$ *or* $C\ell(B)$.
2. If $(a_i \mid i \in M')$ *be an orthogonal basis for V, then* $(a_K \mid K \subseteq M',\ K \text{ finite})$ *is a basis for* $C\ell(B, V)$.
3. The Clifford product is independent of selection of orthogonal basis, if the product is constructed in $C\ell(B, V)$.

Proof: 1. We shall prove that to any algebra A over $\mathbb{K}$ and any linear mapping $f : V \to A$ such that $f(x)^2 = B(x, x)\, 1_A$, there exists a unique algebra morphism $F : W \to A$ that extends f.

Therefore define a linear mapping $F : W \to A$ necessarily by $F(e_\emptyset) = 1_A$, and $F(e_K) = \Pi_{k \in K}\, f(e_k)$.

For $i \neq j$ and $x = e_i + e_j$, we get $f(x)^2 = B(x, x)\, 1_A \Rightarrow f(e_i)\, f(e_j) = -f(e_j)\, f(e_i)$ by expansion.

Let $H = \{h_1, \ldots, h_p\}$ and $K = \{k_1, \ldots, k_q\}$ with increasing indices. Then $F(e_H)\, F(e_K) = f(e_{h_1}) \ldots f(e_{h_p})\, f(e_{k_1}) \ldots f(e_{k_q})$.

If $h_p > k_1$, then $f(e_{k_1})$ is swapped with $f(e_{h_p})$, and so on for decreasing h-indices.

This is compensated with a factor $\alpha(H, k_1)$. Reducing for two identical elements gives a factor $\beta(H, k_1)$.

The same proces is next done with $f(e_{k_2})$, and this continues until $f(e_{k_q})$. The total factor becomes $\sigma(H, K)$.

Thus $F(e_H)\, F(e_K) = \sigma(H, K)\, F(e_{H \triangle K}) = F(e_H\, e_K)$.

2. Construct a new Clifford algebra $C\ell(B, V)'$ from the basis (a_i). Then $(a_K \mid K \subseteq M',\ K \text{ finite})$ is a basis for $C\ell(B, V)'$.

By corollary 1.3 there exists a unique algebra isomorphism $F : C\ell(B, V)' \to C\ell(B, V)$ fixing V.

Therefore, as $(F(a_K)) = (\Pi_K\, F(a_k)) = (a_K)$, we get $(a_K \mid K \subseteq M',\ K \text{ finite})$ is a basis for $C\ell(B, V)$.

3. The construction of $C\ell(B, V)'$ can be transferred by F to $C\ell(B, V)$. By this isomorphism the

Clifford product in $C\ell(B,V)'$ is transferred to that of $C\ell(B,V)$.
NB: In proofs it is often a simplification to use W for $C\ell(B,V)$

Definition 2.6. Let $\mathbb{R}^{p,q,r}$ *be a real vector space of dimension* $n = p+q+r$ *with a symmetric bilinear-form B that in diagonal form has* (p,q,r) *times* $(1,-1,0)$*'s respectively in that order. To this correspond a Clifford algebra* $\mathbb{R}_{p,q,r}$*. If* $r=0$*, r can be omitted.*
The complex case $\mathbb{C}_{p,r}$ *is defined likewise, but without q and* -1.

Example. (Generators). Let a real universal Clifford algebra is given by generators: $e_1^2 = e_2^2 = -1$, $e_1 e_2 = -e_2 e_1$. It just gives a little extra work to find a suitable bilinearform B. Here in basis $(e_1\, e_2)$ we get B as a diagonal matrix $(-1,-1)$. It turns out that $C\ell(B,\mathbb{R}^2) = \mathbb{R}_{0,2}$, and this is a version of the quaternions, $\mathbb{H}$, with $e_\emptyset = 1$, $e_1 \to i$, $e_2 \to j$, $e_{12} \to k$.

Example. (Non-universality). In the real universal Clifford algebra $\mathbb{R}_{0,3}$, $q = (1+e_{123})/2$ belong to the center, and thus commutes with any other element. Let $\mathcal{I} = \mathbb{R}_{0,3}\, q = q\, \mathbb{R}_{0,3}$. $\mathcal{I}$ is an ideal, as $q^2 = q$ and $\mathbb{R}_{0,3}\,\mathcal{I} = \mathbb{R}_{0,3}\,\mathbb{R}_{0,3}\, q = \mathcal{I}$ and $\mathcal{I}\,\mathbb{R}_{0,3} = q\,\mathbb{R}_{0,3}\,\mathbb{R}_{0,3} = \mathcal{I}$. $\mathcal{I}$ is proper, as $q\,(1 - e_{123})/2) = 0$ and $q \in \mathcal{I}$.
It turns out (theorem 6.6) that $\mathbb{R}_{0,3}/\mathcal{I}$ is a non-universal Clifford algebra. It is also a version of the quaternions with $e_\emptyset = 1$, $e_1 \to i$, $e_2 \to j$, $e_3 \to k$.

The examples show that universality can not be determined from an algebra alone, it also requires knowledge of the underlying vector space or module.

Chapter 3 The Grassmann algebra over *V*

The Grassmann algebra over *V*. Blades

In the case $B = 0$, we have an exterior or a Grassmann algebra $\Lambda(V) = C\ell(0,V)$ where the product is denoted $\wedge$ and named the outer product. Outer products of elements in V is called *blades*, when they are non-zero.

Theorem 3.1. In $\Lambda(V)$ *define the submodule of elements of grade* $r \in \mathbb{Z}$ *by*
$\Lambda_r(V) = \operatorname{span}\{\wedge_{i=1}^{r} a_i \mid a_i \in V\}$ *for* $r \geq 0$ *and otherwise* $\Lambda_r(V) = \{0\}$.
This makes $\Lambda(V)$ *a graded algebra, as obviously* $\Lambda_r(V) \wedge \Lambda_s(V) \subseteq \Lambda_{r+s}(V)$.
Also define $x \to \langle x \rangle_r$*, as the projection on* $\Lambda_r(V)$ *along* $\oplus_{i \neq r} \Lambda_i(V)$*, and* $\langle x \rangle = \langle x \rangle_0$.
Set $\Lambda_{<p}(V) = \oplus_{i<p} \Lambda_i(V)$*, and also* $x \to \langle x \rangle_R = \oplus_{r \in R} \langle x \rangle_r$ *, where R is a subset of* $\mathbb{Z}$.
Then
1. $\Lambda(V) = \oplus_r \Lambda_r(V)$ *and* $\Lambda_r(V) \wedge \Lambda_s(V) = \Lambda_{r+s}(V)$.
   If $|M|$ *is finite, then* $\operatorname{rank}(\Lambda_{|M|}(V)) = 1$ *and* $\Lambda_r(V) = 0$ *for* $r > |M|$
2. $x \wedge x = 0$ *and* $x_1 \wedge x_2 = -x_2 \wedge x_1$

3. $x_1 \wedge x_2 \wedge \ldots \wedge x_p$ *is multilinear and alternating in the x-variables*

NB: 0 is associated with any grade.

Proof: 1. In $\Lambda(V)$ define $\Omega_r = \operatorname{span}\{e_K \mid |K| = r, K \in \mathcal{F}\}$ for $r \geq 0$ and otherwise $\Omega_r = \{0\}$. Then clearly $\Omega_r \subseteq \Lambda_r$

$\Lambda(V) = \oplus_r \Omega_r$ and $\Omega_r \wedge \Omega_s = \Omega_{r+s}$, which implies $\wedge_{i=1}^{r} \Omega_1 = \Omega_r$ and therefore $\wedge_{i=1}^{r} a_i \in \Omega_r$. Thus $\Omega_r = \Lambda_r$.

If $|M|$ is finite, then $\Omega_{|M|} = \{\mathbb{K}\, e_M\}$ and $r > |M| \Rightarrow \Omega_r = \{0\}$

2. In $x \wedge x = B(x, x) = 0$ set $x = x_1 + x_2$.

3. The expression is multilinear by definition. Alternating means it is 0, if two arguments are identical. If $x_i = x_j$, then by swapping neighbors $x_k = x_{k+1}$ can be obtained, and by (2) the product is 0.

Example: Let $\mathbb{K} = \mathbb{Z}$, $M = \{1\}$ and $x_1 = 2\, e_1$. Then x_1 is linear independent, but can not be extended to a basis.

Theorem 3.2. (Extension by outermorphism). A $\mathbb{K}$*-linear mapping* $f : V_1 \to V_2$ *has a unique extension,* $f_\wedge : \Lambda(V_1) \to \Lambda(V_2)$ *to an algebra morphism, which is grade preserving. Moreover* $f_\wedge$ *is bijective, if f is.*

Proof: As $f(x) \wedge f(x) = 0$, the assertion follows from universal extension, which is grade preserving, as a Grassmann algebra morphism.

Universality, basis and rank

In the next theorem the work is done with Grassmann algebras to ensure the arbitrary base is orthogonal so the construction is possible.

Theorem 3.3 (The Invariant basis property). Two bases for V have the same finite size or are both infinite.

Proof: Construct Grassmann algebras $\Lambda(V)$ and $\Lambda'(V)$ from the bases $(e_i \mid i \in M)$ and $(a_i \mid i \in M')$.
From theorem 2.5 $(e_K \mid K \in \mathcal{F})$ and $(a_{\wedge K} \mid K \subseteq M',\ K \text{ finite})$ are bases for $\Lambda(V)$ and $\Lambda'(V)$.
From theorem 3.2 by extension of $\mathrm{id}_V : V \to V$ follows $f_\wedge$ gives a linear isomorphism $\Lambda_r(V) \to \Lambda'_r(V)$.
If $|M|$ is finite, then $|M| = \max\{r \mid \Lambda_r(V) \neq \{0\}\} = \max\{r \mid \Lambda'_r(V) \neq \{0\}\} = |M'|$, and likewise if $|M'|$ is finite.

Theorem 3.4. Assume M is finite. Then with respect to any basis

1. $|M| = \text{rank}(V)$ *and* $|M| = \max\{r \mid \Lambda_r(V) \neq 0\}$
2. $\Lambda_{|M|}(V) = \mathbb{K}\, e_M$
3. $\text{rank}(\Lambda_r(V)) = \binom{|M|}{r}$
4. $\text{rank}(\Lambda(V)) = 2^{|M|}$ *and* $\text{rank}(\Lambda(V)^+) = \text{rank}(\Lambda(V)^-) = 2^{|M|-1}$

Proof: Follows from theorem 3.1 (1,2) .
3. As $\{e_K \mid |K| = r\}$ is a basis for $\Lambda_r(V)$, it has size equal to the number of subsets of M with r elements.
4. Apply the binomial formula to $(1+1)^{|M|}$ and $(1-1)^{|M|}$. Then the first gives (4a), the sum (4b) and the difference (4c).

Theorem 3.5. From $C\ell(B, V)$ any non isomorphic Clifford algebra A in $\mathcal{A}(V, B)$ can be found as a quotient $C\ell(B, V) / \mathcal{I}$ with ideal $\mathcal{I} \neq \{0\}$.
If M is finite, then *A in $\mathcal{A}(V, B)$ is non-universal* $\Leftrightarrow \text{rank}(A) = 2^k$ and $k < |M|$.

Proof: Define by universality a morphism $F : C\ell(B, V) \to A$. Here F is surjective since A is generated by V.
Thus $A \simeq C\ell(B, V) / \mathcal{I}$ where $\mathcal{I} \neq \{0\}$, as otherwise F is an isomorphism. In the finite case $\text{rank}(A) < 2^{|M|}$ and a divisor in this number.

The Geometric algebra $\mathcal{G}(B, V)$

By making the constructions in $C\ell(B, V)$, it is possible to work with several different Clifford algebras all in the same space.
A universal Clifford algebras $C\ell(B, V)$ can in this way always be supplemented with a Grassmann algebra $\Lambda(V)$.

Definition 3.6. The geometric algebra $\mathcal{G}(B, V)$ or $\mathcal{G}(V)$ is the double algebra of $C\ell(B, V)$ and $\Lambda(V)$ in the same space.
For $x_i \in V$ set $x_I = \Pi_{i \in I, \uparrow}\, x_i$ and $x_{\wedge I} = \wedge_{i \in I, \uparrow}\, x_i$. By construction $e_{\wedge I} = e_I$.

We may silently consider $\mathbb{R}_{p,q,r}$ and $\mathbb{C}_{p,q,r}$ extended to geometric algebras. If r is omitted, then $r = 0$.

Chapter 4 Morphisms

Anti-morphisms

Definition 4.1. To every algebra A is in the same linear space associated an opposite algebra A^o

with multiplication $X \circ Y = Y X$.
The linear identity $A \to A^o$*is an anti-automorphism, and is also denoted* $o : A \to A^o$. *Moreover* $A^{oo} = A$ *and* $o^2 = \mathrm{id}_A$.
That this multiplication makes A^o an algebra is easily verified, and also that $A^{oo} = A$ and $o^2 = \mathrm{id}_A$.

Theorem 4.2. For any algebra A over $\mathbb{K}$ *and any linear mapping* $f : V \to A$ *such that* $f(x)^2 = B(x, x)\, 1_A$, *there exists a unique algebra anti-morphism* $F^o : C\ell(B, V) \to A$, *which extends f. This extension is also called the universal anti-extension.*

Proof: Let F be the universal extension of f. Clearly $F^o = F \circ o$ is a solution, and F^o is unique, as $F^o \circ o$ is the universal extension of f.
This idea also proves the corollary.

Now the following is obvious.
Corollary 4.3. Assume algebras U_i *belongs to* $\mathcal{A}(V_i, B_i)$ *and* $f_i : V_i \to V_{i+1}$ *is* $\mathbb{K}$*-linear.*
Then f_i *has a unique extension,* $U_i \to U_{i+1}$ *to an algebra anti-morphism, which is an anti-isomorphism, if* f_i *is bijective.*
Let $F_i : U_i \to U_{i+1}$*be a morphism or an anti-morphism and* $F = F_k \circ \ldots \circ F_1$. *If the number of anti-morphism in the composition is odd, then F is anti-morphism, and otherwise a morphism.*

The three main commuting involutions.

Definition 4.4. Let U be a Clifford algebra in $\mathcal{A}(V, B)$, *not necessarily universal.*
As proved a linear mapping $f : V \to V$ *has at most one extension to an automorphism or anti-automorphism of U.*
If they exists,
the main or grade automorphism $X \to \hat{X}$ *is the extension of* $f : V \to V$, $f(x) = -x$ *to an automorphism of U.*
the reversion $X \to \tilde{X}$ *is the extension of* $f : V \to V$, $f(x) = x$ *to an anti-automorphism of U.*
the Clifford conjugation $X \to \overline{X}$ *is the extension of* $f : V \to V$, $f(x) = -x$ *to an anti-automorphism of U.*

Theorem 4.5. $C\ell_V(B)$ *is extended to a geometric algebra to make the grade concept available.*
1. In $C\ell_V(B)$ *the main automorphism, the reversion, and the Clifford conjugation exists.*
2. For the main automorphism holds $\mathrm{grade}(X) = r \Rightarrow \hat{X} = (-1)^r X$
3. For the reversion holds $(X Y)^\sim = \tilde{Y}\tilde{X}$, $(a_1 a_2 \ldots a_r)^\sim = a_r \ldots a_2 a_1$ *for* $a_i \in V$ *and* $\mathrm{grade}(X) = r \Rightarrow \tilde{X} = (-1)^{r(r-1)/2} X$

4. For the Clifford conjugation holds $\overline{X} = \hat{X}^{\sim}$ *and* $\text{grade}(X) = r \Rightarrow \overline{X} = (-1)^{r(r+1)/2} X$

5. These three mappings are grade preserving, involutions, commuting and independent of the B. Each one is the composition of the two others.

Proof: 1. Universality secures the existence.

2. As f^2 is the identity, the extension is bijective. In the standard basis for W we get $\hat{e_K} = \Pi_K (-e_k) = (-1)^{|K|} e_K$.

3. In the standard basis for W by swapping $(|K|-1) + \ldots + 2 + 1$ neighbors staring from one end we get $\tilde{e_K} = \Pi_{k \in K, \downarrow} e_k = (-1)^{|K|(|K|-1)/2} e_K$.

4. Follows from corollary 4.3.

5. Obvious from the graded expressions.

NB: Complex conjugation denoted X^{conj} to distinguish it from Clifford conjugation.

Chapter 5 Basic structure of Geometric algebra

Geometric algebra basic formula collection

Definition 5.1. Set $\chi_S = 1$, *if the proposition* S *is true, and else zero. Define in* $\mathcal{G}(B, V)$ *compositions* $\cdot, \rfloor$ *and* $\lfloor$ *by bilinearity by*

$e_H \cdot e_J = \chi_{H=J}\ e_H e_J$, *the scalar product*,

$e_H \rfloor e_J = \chi_{H \subseteq J}\ e_H e_J$, *the left contraction*,

$e_H \lfloor e_J = \chi_{H \supseteq J}\ e_H e_J$, *the right contraction*.

We already know from the product definition based on the α *and* β *functions that*

$e_H \wedge e_J = \chi_{H \cap J = \emptyset}\ e_H e_J$

NB: In case of $\mathbb{K} = \mathbb{C}$ the scalar product is $\mathbb{C}$-bilinear, not hermitian.

Theorem 5.2. In a geometric algebra $\mathcal{G}(B, V)$ *holds*

1. $x X = x \rfloor X + x \wedge X$ *and* $X x = X \wedge x + X \lfloor x$

   $x \cdot y = B(x, y)\, 1_{\mathcal{G}}$,

2. *If* $\text{grade}(X) = r$ *and* $\text{grade}(Y) = s$, *then*

   $X \cdot Y = \langle X Y \rangle = \langle Y X \rangle = Y \cdot X$, $X \rfloor Y = \langle X Y \rangle_{s-r}$, $X \lfloor Y = \langle X Y \rangle_{r-s}$ *and* $X \wedge Y = \langle X Y \rangle_{r+s}$

   The three main involutions are symmetric, $\hat{X} \cdot Y = X \cdot \hat{Y}$, $\tilde{X} \cdot Y = X \cdot \tilde{Y}$, $\overline{X} \cdot Y = X \cdot \overline{Y}$

3. *If* $\text{grade}(X) = r$ *and* $\text{grade}(Y) = s$, *then*

   $r \neq s \Rightarrow X \cdot Y = 0$, $Y \lfloor X = \left(\tilde{X} \rfloor \tilde{Y}\right)^{\sim} = (-1)^{(s+1)\, r} X \rfloor Y$ *and* $X Y = \Sigma_{i=|r-s|\ \text{step}\ 2}^{r+s} \langle X Y \rangle_i$

4. $(X \wedge Y) \rfloor Z = X \rfloor (Y \rfloor Z)$ *and* $(X \wedge Y) \cdot Z = X \cdot (Y \rfloor Z)$

5. $x \rfloor (X Y) = (x \rfloor X)\, Y + \hat{X}\, (x \rfloor Y)$

$x \wedge (X\,Y) = (x \rfloor X)\,Y + \hat{X}\,(x \wedge Y)$
$x \wedge (X\,Y) = (x \wedge X)\,Y - \hat{X}\,(x \rfloor Y)$
$x \rfloor (X\,Y) = (x \wedge X)\,Y - \hat{X}\,(x \wedge Y)$

6. $x \rfloor (X \wedge Y) = (x \rfloor X) \wedge Y + \hat{X} \wedge (x \rfloor Y)$
$x \wedge (X \rfloor Y) = (x \rfloor X) \rfloor Y + \hat{X} \rfloor (x \wedge Y)$

7. $x \rfloor (x_1\, x_2 \ldots x_p) = \sum_{k=1}^{p} (-1)^{k-1}\, x_1\, x_2 \ldots (x \rfloor x_k) \ldots x_p$

8. $x \rfloor (x_1 \wedge x_2 \wedge \ldots \wedge x_p) = \sum_{k=1}^{p} (-1)^{k-1}\, x_1 \wedge x_2 \ldots \wedge\ (x \rfloor x_k) \ldots \wedge x_p$

9. *$x_1, x_2, \ldots, x_p$ are pairwise orthogonal* $\Rightarrow \Pi_{i=1}^{p}\, x_i = \wedge_{i=1}^{p}\, x_i$

10. $x\,X - \hat{X}\,x = 2\,x \rfloor X$ *and* $x\,X + \hat{X}\,x = 2\,x \wedge X$

11. $\forall_{x \in V}\,(x \wedge A = 0) \Leftrightarrow (A \in \mathbb{K}\,e_M,$ *if* $|M|$ *is finite, and otherwise* $A = 0)$.
Assume $\mathbb{K}$ *is a field or the weaker condition* $\mu\,e_i^2 = 0 \Rightarrow (\mu = 0 \text{ or } e_i^2 = 0)$ for $i \in M,\ \mu \in \mathbb{K}$.
Then
$\forall_{x \in V}\,(x \rfloor A = 0) \Leftrightarrow A \in \mathcal{G}(V_0)$, *where* V_0 *is the radical or kernel of* B. (*NB: Always* $\mathbb{K} \subseteq \mathcal{G}(V_0)$)

12. $(x_1 \wedge x_2 \ldots \wedge x_r) \cdot (y_r \wedge \ldots \wedge y_2 \wedge y_1) = \Sigma_\sigma\, s_\sigma\,(x_1 \cdot y_{\sigma(1)}) \ldots (x_r \cdot y_{\sigma(r)})$
where summation is over all permutations σ *of* $\{1, \ldots, r\}$.

13. *Factor expansion of* $x_{\wedge K}$*: Let* $X = x_{\wedge K}$ *and* $B \in \Lambda_s(V)$. *If* $\tau_H = \alpha(H, K \backslash H)\,(B \cdot x_{\wedge H}) \in \mathbb{K}$ **), *then*

$$B \rfloor X = \Sigma_{H \subseteq K,\, |H| = s}\ \tau_H\, x_{\wedge K \backslash H}$$

*) *α is from definition 2.1:* $\alpha(H, J) = \Pi\,(-1)$ for $(h, j) \in H \times J$ and $h > j$

Proof:

By linearity it is sufficient to sketch proofs of the statements for basis elements.

As we may assume $x = e_i$, $X = e_H \in \Lambda_r$, $Y = e_J \in \Lambda_s$, $Z = e_K \in \Lambda_t$, we get

1. (1a) $h \in H \Rightarrow e_h \wedge e_H + e_h \rfloor e_H = e_h\ e_H$ and similar for $h \notin H$. (1b) Likewise.
   (1c) $e_i \rfloor e_j = B(e_i, e_j)\, 1_{\mathcal{G}}$ is obvious in the two cases $i = j$ and $i \neq j$
2. (2a) $e_H \cdot e_J = \chi_{H=J}\ e_H\, e_J = e_J \cdot e_H$ and $H \neq J \Rightarrow\ e_H\, e_J$ is not a scalar.
   (2b) $e_H \rfloor e_J = \chi_{H \subseteq J}\, e_H\, e_J = \chi_{H \subseteq J}\, e_{H \triangle J} = \chi_{H \subseteq J}\, e_{J \backslash H}$ which has grade $s - r$.
   (2c) Like (2b)
   (2d) $e_H \wedge e_J = \chi_{H \cap J = \emptyset}\ e_H\, e_J$ which has grade $s + r$.
   (2e) If $r \neq s$ it is obvious. Otherwise clear from graded expressions.
3. (3a) Follows from the definition of the scalar product
   (3b) Set $\tau_K = (-1)^{|K|\,(|K|-1)/2}$, as temporary set function.
   $(e_H \lfloor e_J)^\sim = \chi_{J \subseteq H}\,(e_H\, e_J)^\sim = \chi_{J \subseteq H}\, \tilde{e_J}\, \tilde{e_H} = \tau_J\, \tau_H\, \chi_{J \subseteq H}\, e_J\, e_H = \tau_J\, \tau_H\, e_J \rfloor e_H = \tilde{e_J} \rfloor \tilde{e_H}$
   and with some work $\tau_J\, \tau_H\, \tau_{H \backslash J} = (-1)^{(s+1)\,r}$.
   (3c) From a Venn-diagram obviously $|H \triangle J| = |J| - |H| + 2\,|H \backslash J|$. Thus $|H \triangle J| = |H| - |J| + 2\,|J \backslash H|$.

Hence the grade of $e_H\, e_J$ is an even number or zero higher than $||J| - |H||$. The upper limit comes from $e_H \wedge e_J$.

4. (4a) We have $\chi_{H\cap J=\emptyset}\,\chi_{H\cup J\subseteq K} = \chi_{H\subseteq(K\setminus J)}\,\chi_{J\subseteq K}$, as both represents the situation: K including disjunct H and J.

Multiplying the equation with $e_H\, e_J\, e_K$ gives $(e_H \wedge e_J)\rfloor e_K = e_H\rfloor(e_J \rfloor e_K)$.

(4b) Likewise $\chi_{H\cap J=\emptyset}\,\chi_{H\cup J=K} = \chi_{H=(K\setminus J)}\,\chi_{J\subseteq K}$, as both represents the situation: K equal to union of disjunct H and J.

Multiplying the equation with $e_H\, e_J\, e_K$ gives $(e_H \wedge e_J)\cdot e_K = e_H \cdot (e_J \rfloor e_K)$.

5. (5a) The equation $e_i \rfloor (e_H\, e_J) = (e_i \rfloor e_H)\, e_J + \hat{e}_H\,(e_i \rfloor e_J)$ can be reduce to

1. If $i \in H \cap J$: $0 = e_i\, e_H\, e_J + \hat{e}_H\, e_i\ e_J$, as $e_H\, e_J = \sigma(H, J)\, e_{H \triangle J}$ and $\hat{e}_H\, e_i = -e_i\, e_H$
2. If $i \in H\setminus J$: $e_i\, e_H\, e_J = e_i\, e_H\, e_J + 0$
3. If $i \in J\setminus H$: $e_i\, e_H\, e_J = 0 + \hat{e}_H\, e_i\ e_J$, as $\hat{e}_H\, e_i = e_i\, e_H$
4. If $i \notin H \cup J$: $0 = 0 + 0$

(5b) The equation $e_i \wedge (e_H\, e_J) = (e_i \rfloor e_H)\, e_J + \hat{e}_H\,(e_i \wedge e_J)$ can be reduce to

1. If $i \in H \cap J$: $e_i\, e_H\, e_J = e_i\, e_H\, e_J + 0$
2. If $i \in H\setminus J$: $0 = e_i\, e_H\, e_J + \hat{e}_H\, e_i\ e_J$ and $\hat{e}_H\, e_i = -e_i\, e_H$
3. If $i \in J\setminus H$: $0 = 0 + 0$
4. If $i \notin H \cup J$: $e_i\, e_H\, e_J = 0 + \hat{e}_H\, e_i\ e_J$ and $\hat{e}_H\, e_i = e_i\, e_H$

(5c) From $x\,(X\,Y) = (x\,X)\,Y$ subtract (5a) and use (1)

(5d) From $x\,(X\,Y) = (x\,X)\,Y$ subtract (5b) and use (1)

6. If grade(X) = r and grade(Y) = s, then for (6a) take grade $r + s - 1$ in (5a) or (5d), and for (6b) take grade $s - r + 1$ in (5b) or (5c)

7,8. Follows from (5a, 6a) with induction step, as e.g. $x\rfloor(x_1(x_2 \ldots x_p)) = (x \rfloor x_1)\,(x_2 \ldots x_p) - x_1(x \rfloor (x_2 \ldots x_p))$

9. Follows from (1a, 7) with induction step, as e.g. $x_1\, x_2 \ldots x_p = x_1 \wedge (x_2 \ldots x_p) + x_1\rfloor(x_2 \ldots x_p) = x_1 \wedge (x_2 \ldots x_p)$

10. From (1) and (3) follows $\hat{X}\,x = \hat{X} \wedge x + \hat{X} \lfloor x = x \wedge X - x\rfloor X$. This and (1a) gives the assertion.

11. The radical $V_0 = \text{span}\left(\{e_i \,|\, e_i^2 = 0\,,\, i \in M\}\right)$. Let $A = \Sigma\, \mu_K\, e_K$ in the standard basis. Then

(11a) If $x \wedge A = 0$, then $e_h \wedge A = \Sigma\, \mu_K\, e_{K\cup\{h\}} = 0$ summing over $\{K \,|\, h \notin K\}$.

As $e_{K\cup\{h\}}$ in the sum are different, we have $h \notin K \Rightarrow \mu_K = 0$

(11b) If $x\rfloor A = 0$, then $e_h\rfloor A = \Sigma\ \pm\mu_K\, e_h^2\, e_{K\setminus\{h\}} = 0$ summing over $\{K \,|\, h \in K\}$.

As $e_{K\setminus\{h\}}$ in the sum are different, if ($h \in K$ and $\mu_K \neq 0$), then $e_h^2\, \mu_K = 0 \Leftrightarrow e_h^2 = 0 \Leftrightarrow e_h \in \mathcal{G}(V_0)$.

12. $(x_1 \wedge x_2 \ldots \wedge x_r)\cdot(y_r \wedge \ldots \wedge y_2 \wedge y_1) = \Sigma_\sigma\, s_\sigma\,(x_1 \cdot y_{\sigma(1)}) \ldots (x_r \cdot y_{\sigma(r)})$ for some sign factors s_σ, since by (8) each x is paired with each y once. This is done systematically permuting the y-set with σ, and then contracting successively each x for descending x-indices with the nearest remaining y. If the permutation σ requires k neighbor swaps, the sign change is $s_\sigma = (-1)^k$, which is the sign of the

permutation, $\mathrm{sign}(\sigma)$.
13. If $s > k$, the formula clearly becomes $0 = 0$.
Otherwise assuming $B = b_1 \wedge b_2 \wedge \ldots \wedge b_s$ we get from (4) that $B \rfloor X = b_1 \rfloor (b_2 \rfloor \ldots \rfloor (b_s \rfloor X) \ldots)$.
Equation (7) applied s times gives $k(k-1)\ldots(k-s+1)$ terms of form $\Pi_{k=1}^{s} \pm (b_k \rfloor x_{h_k})\, x_{\wedge K \backslash H}$, where $H = \bigcup_{k=1}^{s} \{h_k\}$.
Hence $B \rfloor X = \Sigma_{H \subseteq K, |H|=s}\ \tau_H\, x_{\wedge K \backslash H}$ with some scalar factor τ_H.
Focusing here on the terms with $x_{\wedge K\backslash H}$ in $B \rfloor X = \alpha(H, K\backslash H)\,(B \rfloor (x_{\wedge H} \wedge x_{\wedge K \backslash H})\ )$, as $x_{\wedge K\backslash H}$ does not influence the ± 1 factors in(7), now follows $\tau_H = \alpha(H, K\backslash H)\,(B \rfloor x_{\wedge H}) = \alpha(H, K\backslash H)\,(B \cdot x_{\wedge H})$.

Determinant Theorem 5.3. Let $f : V \to V$ be linear mapping.
1. If V has a finite basis, then the determinant $\det(f)$ *is defined by* $F(e_M) = \det(f)\, e_M$ *independent of basis.*
2. Moreover $(f \circ g)_\wedge = f_\wedge \circ g_\wedge$, $\det(f \circ g) = \det(g)\det(f)$*, and* $\det(f^{-1})\det(f) = 1$ *when f is bijective.*
3. Assume $m = |M|$, $M = \{1, \ldots, m\}$ and $f(a_s) = \Sigma_i\, \theta_s^i\, a_i$ in some basis $(a_i \mid i \in M)$. Then $\det(f) = \Sigma_\sigma\, \mathrm{sign}(\sigma)\, (\,\theta^1_{\sigma(1)} \ldots \theta^m_{\sigma(m)}\,)$*, where summation is over all permutations σ of M.*

Proof: 1. Let $(a_i \mid i \in M)$ be another basis. As $\Lambda_{|M|}(V) = \mathbb{K}\, e_M$, we get $a_M = \lambda\, e_M$ and $f_\wedge(a_M) = \det(f)\, a_M$ from the definition.
2. Equality of $f_\wedge \circ g_\wedge$ and $(f \circ g)_\wedge$ follow from uniqueness of universal extension.
From $\det(f \circ g)\, e_M = (f \circ g)_\wedge\, (e_M) = f_\wedge(g_\wedge(e_M)) = f_\wedge(\det(g)\, e_M) = \det(g)\det(f)\, e_M$ (b, c) follows.
3. Introduce a Clifford algebra by letting $(a_i \mid i \in M)$ be orthonormal.
Then $\theta_s^i = a_i \cdot f(a_s)$ and $\det(f) = e_M \cdot F(e_M)^\sim$, and the assertion follows from (12).

Automorphism Theorem 5.4. Let $f : V \to V$ be linear mapping, such that $f(x)^2 = B(x, x)\, 1_A$. Then f has two universal extensions:
To an outermorphism $f_\wedge : \Lambda(V) \to \Lambda(V)$, and to Clifford algebra isomorphism $F : C\ell_V(B) \to C\ell_V(B)$.
Assume $B(f(x), f(y)) = B(x, y)$, or $\mathbb{K}$ is a field not of characteristic 2. Then $f_\wedge$ is called the universal extension of f to $\mathcal{G}(B, V)$, as
$F = f_\wedge$, and thus is an outermorphism, and furthermore grade preserving, an orthogonal isomorphism, an isomorphism for $\rfloor$ and $\lfloor$, and commutes with the three main involutions.

Proof: For $i \neq j$ and $x = e_i + e_j$, we get $f(x)^2 = B(x, x)\, 1_A \Rightarrow f(e_i)\, f(e_j) + f(e_j)\, f(e_i) = 0$ by expansion. From theorem 5.2 (1) now follows
$0 =$.

$$f(e_i)\, f(e_j) + f(e_j)\, f(e_i) = f(e_i) \wedge f(e_j) + f(e_i) \cdot f(e_j) + f(e_j) \wedge f(e_i) + f(e_j) \cdot f(e_i) = 2\, f(e_i) \cdot f(e_j)$$

Hence $(f(e_k) \mid k \in K)$ are pairwise orthogonal, and by theorem 5.2 (9) $f_\wedge(e_K) = \wedge_{k \in K}\ f(e_k) = \Pi_{k \in K}\, f(e_k) = F(e_K)$, and by linearity $F = f_\wedge$.

Thus F is grade preserving, also expressed as $F(\langle Z\rangle_k) = \langle F(Z)\rangle_k$. Let grade($X$) = r and grade(Y) = s. Then from theorem 5.2 (2d) follows

$$F(X \wedge Y) = F(\langle X\,Y\rangle_{r+s})) = \langle F(X\,Y)\rangle_{r+s} = \langle F(X)\,F(Y)\rangle_{r+s} = F(X) \wedge F(Y)$$

Likewise can the other compositions be treated with use of theorem 5.2.2, e.g.

$$F(X \cdot Y) = F(\langle X\,Y\rangle)) = \langle F(X\,Y)\rangle = \langle F(X)\,F(Y)\rangle = F(X) \cdot F(Y),$$

$$F(X \rfloor Y) = F(\langle X\,Y\rangle_{s-r})) = \langle F(X\,Y)\rangle_{s-r} = \langle F(X)\,F(Y)\rangle_{s-r} = F(X) \rfloor F(Y)$$

The involution statement follow from the explicit graded expressions.

Anti-automorphism Theorem 5.5. Define $G^{\sim}(X) = G(X^{\sim})$. Let $f : V \to V$ be linear mapping, such that $f(x)^2 = B(x, x)\,1_A$. Then f has two unique universal anti-extensions: To an anti-outermorphism $f_\wedge^{\sim} : \Lambda(V) \to \Lambda(V)$, and to Clifford algebra anti-isomorphism $F^{\sim} : C\ell_V(B) \to C\ell_V(B)$.
Assume $B(f(x), f(y)) = B(x, y)$, or $\mathbb{K}$ is a field not of characteristic 2. Then $f_\wedge^{\sim}$ is called the universal anti-extension of f to $\mathcal{G}(B, V)$, as
1. $F^{\sim} = f_\wedge^{\sim}$, and thus is an anti-outermorphism, grade preserving, an orthogonal isomorphism and commutes with the three main involutions. Furthermore $F^{\sim}(X \rfloor Y) = F^{\sim}(Y) \lfloor F^{\sim}(X)$ and $F^{\sim}(Y \lfloor X) = F^{\sim}(X) \rfloor F^{\sim}(Y)$.
2. If V has a finite basis, then $F^{\sim}(e_M) = (-1)^{|M|\,(|M|-1)/2} \det(f)\, e_M$.

Proof: 1. Using the Automorphism Theorem and reversion all becomes obvious. E.g. if grade(X) = r and grade(Y) = s, then
$F^{\sim}(X \rfloor Y) = F^{\sim}(\langle X\,Y\rangle_{s-r}) = F\left(\left\langle \tilde{Y}\,\tilde{X}\right\rangle_{s-r}\right)\right) = \left\langle F(\tilde{Y})\,F(\tilde{X})\right\rangle_{s-r} = F(\tilde{Y}) \lfloor F(\tilde{X}) = F^{\sim}(Y) \lfloor F^{\sim}(X)$, and likewise for $\lfloor$ and the inner product.
2. $F^{\sim}(e_M) = \det(f)\,\tilde{e_M} = (-1)^{|M|\,(|M|-1)/2} \det(f)\, e_M$

Examples. The three main involutions.

Theorem 5.6. Assume B is regular and $\mathbb{K}$ is a field of characteristic $\neq 2$. Then

A is universal in $\mathcal{A}(V, B)$ $\Leftrightarrow$ A has a main automorphism

Proof: The way $\Rightarrow$ has been proved, so assume A is non-universal and a main automorphism $\psi : A \to A$ exists. Thus $\psi(x) = -x$.
From universality of $C\ell_V(B)$, we get a unique algebra morphism $F : C\ell_V(B) \to A$, such that $x^2 = F(x)^2$, $x \in V$, and with kernel ideal $\mathcal{I} \neq \{0\}$.
Also from B is regular follows $\mathcal{I} \cap V = \emptyset$.
As $x \to \psi\left(F(\hat{x})\right) = \psi(-F(x)) = F(x)$, the universal extension of this gives $\psi\left(F(\hat{X})\right) = F(X)$.
Hence $\psi\left(F(\hat{\mathcal{I}})\right) = F(\mathcal{I}) = \{0\} \Rightarrow F(\hat{\mathcal{I}}) = \{0\}$, and therefore $\hat{\mathcal{I}} \subseteq \mathcal{I}$
Choose $X \in \mathcal{I} \backslash \{0\}$ with lowest highest grade term. Then $X_0 = \left(X + \hat{X}\right) / 2 \in \mathcal{I}$ and

$X_1 = \left(X - \hat{X}\right)/2 \in \mathcal{I}$.

We have $x\,X_r \in \mathcal{I}$ and $X_r\,x \in \mathcal{I}$, and theorem 5.2 (10a) gives $2\,x\rfloor X_r \in \mathcal{I}$ for $r = 0$, 1.

From $2\,x\rfloor X$ has lower highest grade term than X follows the contradiction $X = 0$, and we are done.

Linear independency in Grassmann algebras

Definition 5.7. *A list of elements in a module is linear independent, if the only (finite) linear combination of the elements giving zero is that with zero factors. Linear dependent means not linear independent.*

Obviously holds: A list of elements is linear independent ⇔ *every finite sublist is linear independent*

Examples. 1) 0 is linear dependent. 2) $x \in V\backslash\{0\}$ is linear independent, iff $\lambda\,x = 0 \Rightarrow \lambda = 0$. 3) A basis is linear independent.

Theorem 5.8. Let H be finite. Then

A: $(x_h \mid h \in H)$ *is linear independent* ⇔ B: $x_{\wedge H}$ *is linear independent* ⇔ C: $(x_{\wedge K} \mid K \subseteq H)$ *is linear independent*

Proof: 1. Assume (A). Define a geometric algebra structure on $\Lambda(V)$ by letting (e_i) be an orthonormal basis and $q(i) = 1$.

Also define a induction proposition $\theta(j) : (x_{\wedge J} \mid J \subseteq H,\ |J| < j)$ is linear independent.

$\theta(j)$ is proved by induction after j. $\theta(0)$ is obvious and $\theta(1)$ holds for $j = 1$ by assumption, so assume $\theta(j)$ holds and $j \geq 1$.

Let $\lambda \neq 0$, $|J| = j$ and $J \subseteq H$. Then $0 \neq \lambda\,x_{\wedge J} = \Sigma_{K\subseteq I}\,\lambda_K\,e_K$ and pick one $\lambda_K \neq 0$.

If $h \in H\backslash J$, then by factor expansion from theorem 5.2.13

$\tilde{e_K}\rfloor(\lambda\,x_{\wedge J} \wedge x_h) = \lambda_K\,x_h + \Sigma_{k\in J}\,\mu_k\,x_k \neq 0$ and thus $\lambda\,x_{\wedge J} \wedge x_h \neq 0$. This proves $\theta(j+1)$.

Now $\theta(|H|)$ follows from the induction principle. This imply (B).

2. Assume not(C), i.e. $X = \Sigma_{i=1}^{m}\,\lambda_i\,x_{\wedge K_i} = 0$, where each $\lambda_i \neq 0$, $K_i \subseteq H$ and $i \neq j \Rightarrow K_i \neq K_j$.

Select k with $x_{\wedge K_k}$ of lowest grade. This imply $K_i \subseteq K_k \Rightarrow K_i = K_k$ and $X \wedge x_{\wedge H\backslash K_k} = \lambda_k\,x_{\wedge H} = 0$, which shows not(B).

3. (A) is a special case of (C)

NB: In (C) it is not only one element, but a list of $2^{|H|}$ elements.

Following corollaries are easy consequences of the theorem:

Corollary 5.9. $S = (x_1, x_2\, \ldots, x_p)$ *is linear independent* ⇔ $S_\wedge = x_1 \wedge x_2 \wedge \ldots \wedge x_p$ *is linear independent*

Corollary 5.10. Allow $H_0 \subseteq M$ to be infinite. Then
$(x_h \mid h \in H_0)$ *is linear independent* $\Leftrightarrow$ $(x_{\wedge K} \mid K \subseteq H_0,\ K \text{ finite})$ *is linear independent*

Inclusion

Many statements from this section on have analogous versions created with obvious use of reversion.

Definition 5.10. If U is a submodule of V, then set $\Lambda(U) = \operatorname{span}\{\wedge_{i=0}^{m} U \mid m \in \mathbb{N}\}$*, which obviously is the Grassmann-algebra generated by U.*
Also set $\Lambda_{>0}(U) = \operatorname{span}\{\wedge_{i=1}^{m} U \mid m \in \mathbb{N}\}$.

Theorem 5.11. Let $A = a_{\wedge H}$ be a blade.
Define modules $V_A = \{x \in V \mid x \wedge A = 0\}$, $V_{A^\perp} = \{x \in V \mid \forall_{h \in H}\ x \cdot a_h = 0\}$ and set $A^{\shortparallel} = \Lambda(V_A)$, $A^{\perp} = \Lambda_{>0}(V_{A^\perp})$.
Obviously $\operatorname{span}\{a_h \mid h \in H\} \subseteq V_A$ *implying* $\Lambda(\{\operatorname{span}\{a_h \mid h \in H\}) \subseteq A^{\shortparallel}$*. Moreover also $V_1 = \{0\}$ and $V_{1^\perp} = V$ and $1^{\shortparallel} = \{0\}$, $1^{\perp} = \Lambda_{>0}(V)$.*
Omitting $\shortparallel$ like in $X \subseteq A$ instead of $X \subseteq A^{\shortparallel}$ can be used, if it is clear that A means an algebra and not a blade.
Inclusions like $X \subseteq A^{\shortparallel}$ or $X \subseteq A^{\perp}$ may be used for elements, as in $e_1 \subseteq A^{\shortparallel}$ meaning $\{e_1\} \subseteq A^{\shortparallel}$
Then

1. $X \in \operatorname{span}\{a_{\wedge H_1}, \dots, a_{\wedge H_k} \mid \forall_{h \in H}\ H_h \subseteq H\} \Rightarrow X \subseteq A^{\shortparallel}$
   NB: The opposite inclusion is true, if V is a vectorspace; but not generally for modules.
2. $B \rfloor A \subseteq A^{\shortparallel}$
3. $C \rfloor A = C\,A$*, when* $C \subseteq A^{\shortparallel}$
4. $A^2 = A \rfloor A = A \cdot A$ *and* A *is invertible* $\Leftrightarrow A \cdot A$ *invertible* $\Rightarrow A^{-1} = A/(A \cdot A)$
5. *Assume A is invertible. Then* $\operatorname{span}\{a_h \mid h \in H\} = V_A$.

NB: In Corollary 6.2.5 is proved: A invertible $\Rightarrow$ $V = V_A \oplus V_{A^\perp}$

Proof:
1. Obvious.
2. Obvious from factor expansion and linearity.
3. By linearity we may assume $C = c_K$ with all parts $c_r \subseteq A^{\shortparallel}$, and proceed by induction after $p = |K|$. It is obvious for $p = 0$, and true for $p = 1$, as by $c_r \rfloor A = c_r\,A - c_r \wedge A = c_r\,A$ from theorem 5.1.1.
Now the induction step.

$$(c_s \wedge c_{\wedge K}) \rfloor A = c_s \rfloor (c_{\wedge K} \rfloor A) = c_s \rfloor (c_{\wedge K} A) = (c_s \rfloor c_{\wedge K})\,A + \hat{c_{\wedge K}}(c_s \rfloor A) = (c_s \rfloor c_{\wedge K})\,A + \hat{c_{\wedge K}}\,c_s\,A =$$

$$(c_s \rfloor c_{\wedge K})\,A + \left(\hat{c_{\wedge K}} \wedge c_s\right)A + \left(\hat{c_{\wedge K}} \lfloor c_s\right)\ A = (c_s \wedge c_{\wedge K})\,A, \text{ as } \left(\hat{c_{\wedge K}} \lfloor c_s\right) = (-1)^{p+1}\left(c_s \rfloor \hat{c_{\wedge K}}\right) = -\left(c_s \rfloor c_{\wedge K}\right)$$

The induction principle finishes this part.
4. Follows from (3) that $A \rfloor A = A\,A$ and the remaining is obvious.
5. Assume $x \in V_A$. Then $x \subseteq A^{||}$, $x\,A = x \rfloor A$ and from factor expansion, theorem 5.2.13, follows $x \rfloor A$ is a linear combination of $(h-1)$-blades of form $a_{\wedge H_i}$. Therefore $x = \Sigma\, \lambda_i\, a_{\wedge H_i}\, A = \Sigma\, \lambda_i (a_{\wedge H_i} \rfloor A)$, and again from factor expansion $x = \Sigma\, \mu_j\, a_j$. Thus $V_A \subseteq \text{span}\,\{a_h \mid h \in H\} \subseteq V_A$.

Examples: 1. Assume a *blade* $C = c_{\wedge K}$. Then $C \in A^{\perp} \Leftrightarrow \forall\, h \in H,\ k \in K\, (c_k \cdot a_h = 0) \Leftrightarrow A \in C^{\perp}$.
2. Clearly $C \in A^{\perp} \Rightarrow C \rfloor A = 0$, but the opposite is not true, as $\exists\, h \in H\, \forall\, k \in K\, (c_k \cdot a_h = 0) \Rightarrow C \rfloor A = 0$ which only needs one element a_h.
3. Let $\mathbb{K} = \mathbb{Z}_9$, $M = \{1, 2, 3\}$, $e_1^2 = 1$, $e_2^2 = 1$, $e_3^2 = 1$, such that B is regular. Set $A = e_1 \wedge 3\, e_2$. Then $V_A = \text{span}\,\{e_1, e_2\}$, though e_2 is not generated by $\{e_1, 3\, e_2\}$.
Furthermore $V_{A^{\perp}} = \text{span}\,\{3\, e_2, e_3\}$..

Lemma 5.12. *Let A be a blade. Then*
1. If $C \subseteq A^{||}$:
$(C \rfloor B)\, A = C \wedge (B\, A)$
$(C \rfloor B) \rfloor A = C \wedge (B \rfloor A)$
$(C \wedge B)\, A = C \rfloor (B\, A)$
$(C\, B) \rfloor A = C\, (B \rfloor A)$
2 . If $C \subseteq \mathbb{K} + A^{\perp}$:
$(C \rfloor B)\, A = C \rfloor (B\, A)$
$(C \rfloor B) \wedge A = C \rfloor (B \wedge A)$
$(C \wedge B)\, A = C \wedge (B\, A)$
$(C\, B\,) \wedge A = C\, (B \wedge A)$

Proof: By linearity, it is sufficient to consider blades. Assume the grades of A, B, C are respectively r, s, t.
The proofs are induction after the grade of $C = c_1 \wedge \ldots \wedge c_t$. Clearly all equations are true for scalar C.
Plenty use is made of theorem 5.2 (1a) and 5.2 (4a).
1. Assume $c \wedge A = 0$ for each $c = c_i$.
$(C \rfloor B)\, A = C \wedge (B\, A)$
1a. Case $r = 1$. From theorem 5.2 (5b): $c \wedge (B\, A) = (c \rfloor B)\, A + \hat{B}\, (c \wedge A) = (c \rfloor B)\, A$, i.e. case $t = 1$. This with $B \to C \rfloor B$ and the induction
assumption gives $((c \wedge C) \rfloor B))\, A = (c \rfloor (C \rfloor B))\, A = c \wedge ((C \rfloor B)\, A) = c \wedge (C \wedge (B\, A))$ and use of the induction principle gives the assertion.
1b. Follows from extracting grade $r - s + t$ in (1a)

1c. From theorem 5.2.5 (d): $c\rfloor(BA) = (c\wedge B)A - \hat{B}(c\wedge A) = (c\wedge B)A$, i.e. case $t = 1$. This with $B \to C\wedge B$ and the induction
assumption gives $(c\wedge(C\wedge B))A = c\rfloor((C\wedge B)A) = c\rfloor(C\rfloor(BA)) = (c\wedge C)\rfloor(BA))$, and the induction principle finishes this part.

1d. To (2b) $(c\rfloor B)\rfloor A = c\wedge(B\rfloor A)$ add $(c\wedge B)\rfloor A = c\rfloor(B\rfloor A)$ to get $(cB)\rfloor A = c(B\rfloor A)$, i.e. case $t = 1$. This with $B \to CB$ and the induction
assumption gives $(cCB)\rfloor A = c((CB)\rfloor A) = c(C(B\rfloor A))$ and also $((c\rfloor C)B)\rfloor A = (c\rfloor C)\ (B\rfloor A)$
By subtraction we get $((c\wedge C)B)\rfloor A = (c\wedge C)\ (B\rfloor A)$, and the induction principle leads to the assertion.

2. Assume $C \subseteq A^{\perp}$. Assume $c\rfloor A = 0$ for each $c = c_i$.

2a. Follows from theorem 5.2 (5b) $c\rfloor(BA) = (c\rfloor B)A + \hat{B}(c\rfloor A) = (c\rfloor B)A$, i.e. case $t = 1$. This with $B \to C\rfloor B$ and the induction
assumption gives
$((c\wedge C)\rfloor B)A = (c\rfloor(C\rfloor B))A = (c\rfloor(C\rfloor B))A = c\rfloor((C\rfloor B)A) = c\rfloor(C\rfloor(BA)) = (c\wedge C)\rfloor(BA)$, and the induction principle gives
the assertion.

2b. Follows from extracting grade $r+s-t$ in (2a)

2c. From $(cB)A = c(BA)$ subtract (2a): $(c\rfloor B)A = c\rfloor(BA)$ to get $(c\wedge B)A = c\wedge(BA)$, i.e. case $t = 1$. This with $B \to C\wedge B$ and the
induction assumption gives $(c\wedge C\wedge B)A = c\wedge((C\wedge B)A) = c\wedge(C\wedge(BA))$, and the induction principle gives the assertion.

2d. To $(c\wedge B)\wedge A = c\wedge(B\wedge A)$ add (2b) $(c\rfloor B)\wedge A = c\rfloor(B\wedge A)$ and to get $(cB)\wedge A = c(B\wedge A)$.
This with $B \to CB$ and the induction assumption gives
$(c(CB))\wedge A) = c((CB)\wedge A) = cC(B\wedge A)$ and also $((c\rfloor C)B)\wedge A) = (c\rfloor C)(B\wedge A)$
Subtracting these two equation gives $(c\wedge C)(B\wedge A) = ((c\wedge C)B)\wedge A$, and the induction principle closes the proof.

Chapter 6 Geometric transformations

Projections

Here are four types of projections treated:
Projection P_A on a blade.
Rejection Q_A by a blade.
Projection P^A along a blade.
Projection P_A^B on A along B.

Short writing like $P(X)$, $Q(X)$ is wide used, when it is clear which mapping it stands for.

The standard formula for projection on u is $x = (x \cdot u)\, u / (u \cdot u)$, and this is generalized here.
NB: Building up from this formula with *outermorphism* is natural, but proofs seems easier starting from a general formula.

Theorem 6.1. For a h-blade A assume $\rho = A \cdot A$ is invertible, thus $A^{-1} = \rho^{-1} A$.
Define the projection on A as $P_A(X) = P(X) = (X \rfloor A) \rfloor A^{-1}$.
1. Then P is grade preserving,
2. $P(X) = (X \rfloor A)\, A^{-1}$, $\;P(X) = \rho^{-1} A\, (A \lfloor X) = \rho^{-1} A \lfloor (A \lfloor X)$
3. $X \subseteq A \Rightarrow P(X) = X$, $P(X) \subseteq A$, $P^2(X) = P(X)$
4. $P(\Lambda(V)) = A^{\shortparallel}$, $P(V) = V_A$, $P(A^{\perp}) = \{0\}$.
5. P is symmetric, $X \cdot P(Y) = P(X) \cdot Y$
6. Moreover P is an outermorphism.

Proof: Assume grade(X) = r and grade(Y) = s. Then
1. grade($X \rfloor A$) = $h - r$ and grade($P(X)$ = $h - (h - r) = r$.
2. (2a) $P(X) = (X \rfloor A)\, A^{-1}$, as $(X \rfloor A) \subseteq A$ which imply $P(X) = \rho^{-1}(X \rfloor A)\, A$.
(2b) Also $\rho\, P(X) = (X \rfloor A) \rfloor A = (-1)^{(h+1)(h-r)} A \lfloor (X \rfloor A) = (-1)^{(h+1)(h-r)+(h+1)r} A \lfloor (A \lfloor X) = A \lfloor (A \lfloor X)$
3. From $X \subseteq A$, follows $P(X) = (X \rfloor A)\, A^{-1} = X A A^{-1} = X$.
As $(X \rfloor A) \subseteq A$, the $P(X) = (X \rfloor A)\, A^{-1} \subseteq A$ and $P^2(X) = P(P(X)) = P(X)$ using (3a).
4. (4a) follows from (3a,b)
(4b) follows from P gradepreserving and (4b), as $P(V) = \langle A^{\shortparallel} \rangle_1 = V_A$.
(4c) If $X \subseteq A^{\perp}$, then by lemma 5.12 (2a) $P(X) = (X \rfloor A^{-1})\, A = X \rfloor (A^{-1} A) = X \rfloor 1 = 0$, as $\mathbb{K} \cap A^{\perp} = \emptyset$.
5. As P is grade preserving and elements of different grades are orthogonal, we may assume $r = s$ and get
$X \cdot P(Y) = \langle X\, P(Y) \rangle = \langle X A^{-1} (A \lfloor Y) \rangle = \langle (X \rfloor A^{-1})\, (A \lfloor Y) \rangle = \langle (X \rfloor A^{-1})\, A\, Y \rangle = P(X) \cdot Y$
6. Use is made of the inclusion property theorem 5.12 (1b) several times:
$P(X) \wedge P(Y) = P(X) \wedge ((Y \rfloor A) \rfloor A^{-1}) = (P(X) \rfloor (Y \rfloor A)) \rfloor A^{-1}$ and
$P(X) \rfloor (Y \rfloor A) = (P(X) \wedge Y) \rfloor A = (-1)^{rs} (Y \wedge P(X)) \rfloor A = (-1)^{rs}\, Y \rfloor (P(X) \rfloor A)$
$= (-1)^{rs}\, Y \rfloor (P(X) A) = (-1)^{rs}\, Y \rfloor (X \rfloor A) = (-1)^{rs} (Y \wedge X) \rfloor A = (X \wedge Y) \rfloor A$
Thus $P(X) \wedge P(Y) = P(X \wedge Y)$.

Corollary 6.2. Define the rejection of X by A as $Q_A(X) = Q(X) = X - P_A(X)$. Then
1. $Q_A(x) = A^{-1} \rfloor (A \wedge x) = A^{-1}(A \wedge x) = (x \wedge A)\, A^{-1}$.
2. $Q \circ P = P \circ Q = 0$, $Q^2(X) = Q(X)$ and symmetry, $X \cdot Q(Y) = Q(X) \cdot Y$.
3. Q is gradepreserving, and $\Lambda_r(V) = P(\Lambda_r(V)) \oplus Q(\Lambda_r(V))$

4. $x \in V_{A^\perp} \Rightarrow Q(x) = x,\ \ Q(A^{||}) = \{0\},\ \ Q(V) = V_{A^\perp}$
5. $V = P(V) \oplus Q(V) = V_A \oplus V_{A^\perp}$

NB: $Q(X)$ is not an outermorphism, but this is a commonly used definition. Below is extension of $Q(x)$ by outermorphism covered.

Proof: Let $P = P_A$.
1. Follows from theorem 5.2 (6b), as $\rho\, Q(x) = x \wedge (A \rfloor A) - (x \rfloor A) \rfloor A = \hat{A} \rfloor (x \wedge A) = A \rfloor (A \wedge x) = A\,(A \wedge x)$, as $A \subseteq (A \wedge x)^{||}$.
Theorem 5.2 (3b) gives $A \rfloor (A \wedge x) = (-1)^{h\,(h+2)}\,(A \wedge x) \lfloor A = (x \wedge A) \lfloor A = (x \wedge A)\, A$
2. Obvious from the definition of Q and theorem 6.1 (3c), (5), e.g. $Q \circ P(X) = P(X) - P(P(X)) = 0$.
3. Gradepreserving is obvious from the definition. $\Lambda_r(V) = P(\Lambda_r(V)) + Q(\Lambda_r(V))$, as $P(X) + Q(X) = X$.
If $X = P(Y) = Q(Z)$, then $X = P(X) + Q(X) = P(Q(Z)) + Q(P(Y)) = 0$. Thus $P(\Lambda_r(V)) \bigcap Q(\Lambda_r(V)) = \{0\}$.
4. (4a), $x \in V_{A^\perp} \Rightarrow x = P(x) + Q(x) = Q(x)$, as $P(A^\perp) = \{0\}$.
(4b) $X \subseteq A \Rightarrow Q(X) = X - P(X) = 0$.
(4c) If $A = a_{\wedge H}$, we get $Q(x) \cdot a_i = x \cdot Q(a_i) = 0$. Thus $Q(x) \in V_{A^\perp}$, and $Q(V) \subseteq V_{A^\perp} \subseteq Q(V)$ by (4a).
5. Now obvious.

Corollary 6.3. Define the projection ***along*** *A,* $\mathcal{P}^A$, *as the extension of* $Q_A(x)$ *by outermorphism. Then* $\mathcal{P}^A$ *is grade preserving, and*
1. $\mathcal{P}^A(X) = \mathcal{P}(X) = A^{-1} \rfloor (A \wedge X) = A^{-1}(A \wedge X) = (X \wedge A)\, A^{-1} = (X \wedge A) \lfloor A^{-1} \subseteq A^\perp$
2. $X \subseteq A^\perp \Rightarrow \mathcal{P}(X) = X,\ \ \mathcal{P}(X) \subseteq A^\perp,\ \ \mathcal{P}^2(X) = \mathcal{P}(X),\ \ \mathcal{P}(A^{||}) = \{0\}$, *and* $P_A \circ \mathcal{P}^A = \mathcal{P}^A \circ P_A = 0$
3. S*ymmetry* $X \cdot \mathcal{P}(Z) = \mathcal{P}(X) \cdot Z$

Example: In $\mathbb{R}_{4,0}$ let $X = (e_1 + e_3) \wedge (e_2 + e_4)$ and $A = e_{12}$. Then $Q_A(X) = X - P_A(X) = X - e_{12} = e_{14} - e_{23} + e_{34}$. This is even not a blade, as $Q(X) \wedge Q(X) = -2\, e_{1234}$.
However $\mathcal{P}(X) = e_3 \wedge e_4$.

Proof: Assume $X = x_1 \wedge \ldots \wedge x_r$.
When Q is extended by outermorphism $w_i = P_A(x_i)$ and $y_i = Q_A(x_i) = \mathcal{P}(x_i)$, then $x_i = y_i + w_i$ and $Y = \mathcal{P}(X) = y_1 \wedge \ldots \wedge y_r \subseteq A^\perp$.
1. From $w_i \subseteq A^{||} \Rightarrow A \wedge w_i = 0$ and and lemma 5.12 (2b) follows
$A^{-1} \rfloor (A \wedge X) = A^{-1}(A \wedge (y_1 + w_1) \wedge \ldots \wedge (y_r + w_r)) = \rho^{-1} A \rfloor (A \wedge Y) = \rho^{-1}(A \rfloor A) \wedge Y = Y = \mathcal{P}(X)$.
Finally $A \rfloor (A \wedge X) = (-1)^{(h+r+1)\,h}\,(-1)^{h\,r}\,(X \wedge A) \lfloor A = (X \wedge A) \lfloor A$.
2. Follows easily from corollary 6.2 and $\mathcal{P}$ being an outermorphism.
3. We may assume grade$(Z) = r$. Then

$$X \cdot \mathcal{P}(Z) = \left\langle X\, A^{-1}(A \wedge Z)\right\rangle = \left\langle \left(X \wedge A^{-1}\right)\ (A \wedge Z)\right\rangle = \left\langle \left(X \wedge A^{-1}\right)\ A\, Z\right\rangle = \mathcal{P}(X) \cdot Z$$

Theorem 6.4 Projection P_A^B on A along B (with $(A \wedge B)^{\perp}$ fixed). Assume $\mathbb{K}$ is a field not of characteristic 2.
For a r-blade A and a s-blade B let $C = A \wedge B$ and assume C is invertible and set $\eta = (B \wedge A) \cdot C$. Then $C^{-1} = (-1)^{rs}\, \eta^{-1}\, C$ and
1. $V = V_A \oplus V_B \oplus V_{\perp C}$ as direct sum of vectorspaces, and this defines projections. P_A^B is the projection on $V_A \oplus V_{\perp C}$ along V_B.
2. P_A^B extended by outermorphism gives $P_A^B(X) = \eta^{-1}(A \rfloor C) \rfloor (B \wedge X)$
3. $X \subseteq \Lambda(V_A + V_{\perp C}) \Rightarrow P(X) = X$, $X \subseteq B \Rightarrow P(X) = 0$, $P(X) \subseteq \Lambda(V_A + V_{\perp C})$, $P^2(X) = P(X)$,
$P_A^B \circ P_B^A = P_B^A \circ P_A^B = \mathcal{P}^C$, $P_B^A + P_A^B - \mathcal{P}^C = \mathrm{id}_{\Lambda(V)}$

Proof: As all holds for $r = 0$ or $s = 0$ where $P_1^B(X) = \mathcal{P}^B(X)$ and $P_A^1(X) = X$, assume $r, s > 0$.
1. $V_C \oplus V_{\perp C}$ from corollary 6.2. Let $(a_i, i \in I)$, $(b_j, j \in J)$ be bases for V_A and V_B, such that $A = a_{\wedge I}$ and $B = b_{\wedge J}$.
As $C = a_{\wedge I} \wedge b_{\wedge J} \neq 0$, the combined list of bases is a basis for V_C, and $V_C = V_A \oplus V_B$.
2. Let $X = x_1 \wedge x_2 \wedge \ldots \wedge x_r$, $x_i = a_i + b_i + z_i$ and $y_i = a_i + z_i$, where $a_i \in V_A$, $b_i \in V_B$ and $z_i \in V_{\perp C}$.
$P_A^B(x)$ extended by outermorphism gives $P_A^B(X) = Y = \wedge_i\, y_i$.
Expanded as a sum of blades, $X - Y = \Sigma \ldots \wedge b_k \wedge \ldots$, where each blade contain at least one factor b_k.
Hence $B \wedge (X - Y) = 0$ and $B \wedge X = B \wedge Y$. By factor expansion
$(A \rfloor C) \rfloor (B \wedge Y) = ((A \rfloor C) \rfloor B) \wedge Y + \Sigma$ (terms of form $k_j((A \rfloor C) \cdot (B_j \wedge Y_j)) \wedge B'_j \wedge Y'_j$), where B and Y are split and $\mathrm{grade}(Y_j) \geq 1$. Let $Z = \wedge_i\, z_i$ split like Y. Now $(A \rfloor C) \cdot (B_j \wedge Y_j) = (B_j \wedge Y_j) \rfloor (A \rfloor C) = (B_j \wedge Y_j \wedge A) \rfloor C = \pm((B_j \wedge A) \wedge Z_j) \rfloor C = 0$ by lemma 5.12 (2b) and as $Z \subseteq C^{\perp}$.
Collected we get $(A \rfloor C) \rfloor (B \wedge X) = (A \rfloor C) \rfloor (B \wedge Y) = ((A \rfloor C) \cdot B)\, Y = (B \rfloor (A \rfloor C))\, Y = \eta\, P_A^B(X)$, as $(A \rfloor C) \subseteq Y^{\perp}$ and lemma 5.12 (2b).
3. An easy consequence of (1) for X of grade 1, and then by outermorphism extension.

Example: In $\mathbb{R}_{1,1}$ let $A = e_1 - e_2$, $B = e_1 + e_2$. Then $C = A \wedge B = 2\, e_1 \wedge e_2 = 2\, e_1\, e_2$, which is invertible, but A and B not, as $A^2 = B^2 = 0$.

A formula for the reflection $R_A^B(x)$ in A along B with $(A \wedge B)^{\perp}$ fixed is easily optained as

$$P_A^B(x) - P_B^A(x) = \eta^{-1}(A \rfloor C) \rfloor (B \wedge x) - (-1)^{rs}\, \eta^{-1}(B \rfloor C) \rfloor (A \wedge x)$$

$R_A^B(x)$ can be extented by outermorphism to $R_A^B(X)$; but a simplification is not at hand.

Orthogonal isomorphims. Reflections

In $\mathbb{R}_{3,0}$ the reflection *along* a vector a is $\mathcal{R}^a(x) = Q_a(x) - P_a(x)$ and the reflection *in* a is $R_a(x) = P_a(x) - Q_a(x) = -\mathcal{R}^a(x)$.

As $x = Q_a(x) + P_a(x)$, we get $\mathcal{R}^a(x) = x - 2\,P_a(x) = x - 2\,(x \rfloor a)\,a^{-1} = x - (x\,a - \hat{a}\,x)\,a^{-1} = -a\,x\,a^{-1}$, and this formula will be generalized.

NB: Building up from this formula with *outermorphism* is natural, but proofs seems easier starting from a general formula.

Theorem 6.5. For a h-blade A assume $\rho = A \cdot A$ is invertible, such that $A^{-1} = \rho^{-1} A$.
Define the reflection ***along*** *A by linearity and*

$$\mathcal{R}^A(X) = \mathcal{R}(X) = (-1)^{h\,r}\,A\,X\,A^{-1}, \text{ when } \operatorname{grade}(X) = r. \text{ Then}$$

1. $\mathcal{R}^A(x) = \hat{A}\,x\,A^{-1}$

$\mathcal{R}(x) = x - 2\,P(x)$

$\mathcal{R}(P(x)) = -P(x)$ *and* $\mathcal{R}(Q(x)) = Q(x)$ *justify the words "along A".*

2. Also $\mathcal{R}^2(X) = X$, *and* $\mathcal{R}(X) \cdot \mathcal{R}(Y) = X \cdot Y$.

3. Moreover $\mathcal{R}^A$ is a Clifford algebra isomorphism and universal extension of its restriction to V.
$\mathcal{R}$ is e.g. grade preserving and an outermorphism, as the Automorphism Theorem 5.4 apply to $\mathcal{R}$.

4. Also $\mathcal{R}(P(X)) = P(\hat{X})$, $\mathcal{R}\big(\mathcal{P}^A(X)\big) = \mathcal{P}^A(X)$, *$\mathcal{R}$ is symmetric,* $\mathcal{R}(X) \cdot Y = X \cdot \mathcal{R}(Y)$

5. If V has a finite basis, then $\det\big(\mathcal{R}^A\big) = (-1)^h$

Proof: Assume $\operatorname{grade}(A) = h$, $\operatorname{grade}(X) = r$ and $\operatorname{grade}(Y) = s$. Then

1. 1a. Obvious.

(1b) $\mathcal{R}$ gives a transformation $V \to V$, as $\mathcal{R}(x) = \hat{A}\,x\,A^{-1} = x + \big(-x\,A + \hat{A}\,x\big)\,A^{-1} = x - 2\,(x \rfloor A)\,A^{-1} = x - 2\,P(x) \in V$.

(2b) $\mathcal{R}(Q(x)) = Q(x) - 2\,P(Q(x)) = Q(x)$ and $\mathcal{R}(P(x)) = P(x) - 2\,P(P(x)) = -P(x)$

2. (2a) $\mathcal{R}^2(X) = (-1)^{h\,r}\,A\,\big((-1)^{h\,r}\,A^{-1}\,X\,A\big)\,A^{-1} = X$

(2b) It is clearly zero for $r \neq s$, and for $r = s$ by theorem 5.2 (2a)

$$\mathcal{R}(X) \cdot \mathcal{R}(Y) = (-1)^{h\,r + h\,s}\,\big\langle A\,X\,A^{-1}\,A\,Y\,A^{-1}\big\rangle = \big\langle A\,\big(X\,Y\,A^{-1}\big)\big\rangle = \big\langle\big(X\,Y\,A^{-1}\big)\,A\,\big\rangle = X \cdot Y$$

3. By (1b) $\mathcal{R}$ gives a transformation $V \to V$. Also $\mathcal{R}(1) = 1$.

From (2a) follows $\mathcal{R}$ is bijective and from (2b) that $B(\mathcal{R}(x), \mathcal{R}(y)) = B(x, y)$.

$\mathcal{R}(X)\,\mathcal{R}(Y) = (-1)^{h\,r + h\,s}\,A\,X\,Y\,A^{-1} = \mathcal{R}(X\,Y)$, as $\operatorname{grade}(X\,Y) \equiv r + s \pmod 2$ imply $(-1)^{h\,(r+s)} = (-1)^{h\,\operatorname{grade}(X\,Y)}$, which

shows $\mathcal{R}$ is an Clifford algebra homomorphism. By the uniqueness of the universal extension this proves (3).

4. (4a,b) Follows from uniquenes of extensions to outermorphisms of (1c).

The formula is trivial, if $r \neq s$, and otherwise

$\mathcal{R}(X) \cdot Y = \rho^{-1}(-1)^{h\,r}\,\langle A\,(X\,A\,Y)\,\rangle = \rho^{-1}(-1)^{h\,r}\,\langle (X\,A\,Y)\,A\,\rangle = (-1)^{h\,r - h\,s}\,X \cdot \mathcal{R}(Y) = X \cdot \mathcal{R}(Y)$

5. Counting swappings gives $e_i\,e_M = (-1)^{|M|-1}\,e_M\,e_i$ and $e_H\,e_M = (-1)^{(|M|-1)\,|H|}\,e_M\,e_H$.

Therefore $A\,e_M = (-1)^{(|M|-1)\,h}\,e_M\,A$ and $\mathcal{R}^A(e_M) = (-1)^{h\,|M|}\,A\,e_M\,A^{-1} = (-1)^h\,e_M$

Example. If $|M|$ is finite and e_M invertible, then $(-1)^{|M|}\,e_M\,x\,e_M^{-1} = -x$, and otherwise $x \to -x$ is not a reflection along a blade. However it is a reflection **in** 1.

Corollary 6.6. Define the reflection ***in*** A *as* $R_A(X) = R(X) = \mathcal{R}^A(\hat{X})$. *Then*

1. $R_A(X) = (-1)^{hr} A \hat{X} A^{-1}$ *and, if* $x \in V$*, then*

$R_A(x) = -\mathcal{R}^A(x) = -\hat{A}\, x\, A^{-1}$

$R(x) = -x + 2\, P(x)$

$R(P(x)) = P(x)$ *and* $R(Q(x)) = -Q(x)$ *supports the words "in* A*".*

2. Also $R^2(X) = X$, *and* $R(X) \cdot R(Y) = X \cdot Y$.

3. Moreover R_A *is a Clifford algebra isomorphism and universal extension of its restriction to* V*.*

R is e.g. grade preserving and an outermorphism, as the Automorphism Theorem 5.4 apply to R.

4. $R(P(X)) = P(X)$, $\mathcal{R}\big(\mathcal{P}^A(X)\big) = \mathcal{P}^A(\hat{X})$, $R^2(X) = X$, *R is symmetric,* $R(X) \cdot Y = X \cdot R(Y)$

5. If V *has a finite basis, then* $\det(R_A) = (-1)^{h+|M|}$

NB: $\mathcal{R}^A(X) + R_A(X) = 2\, \langle X \rangle_{\mathrm{even}}$ and not X.

Proof: Obvious from the theorem and earlier results, as e.g. $X \to \hat{X}$ is a symmetric automorphism.

Orthogonal isomorphims. Rotations

Definition 6.7 A versor of order h or a h-versor, $U = u_1 \ldots u_h$ *, as a product of invertible elements* $u_i \in V$*.*

1. As $U^{-1} = u_h^{-1} \ldots u_1^{-1}$ *a versor transformation of* $C\ell(B, V)$ *is defined by linearity and*

$S(X) = S_U(X) = (-1)^{hr}\, U X U^{-1}$ *,when* $\mathrm{grade}(X) = r$*, e.g.* $S(x) = \hat{U}\, x\, U^{-1}$

1. As $S_U = \mathcal{R}^{u_1} \circ \ldots \circ \mathcal{R}^{u_h}$ *the Automorphism Theorem apply to* S_U*, and, if* V *has a finite basis, then* $\det(S_U) = (-1)^h$

2. The Clifford group Γ *is the multiplicative group of versors.*

Define Γ^+*, the rotation versors, as the versors of even order, which obviously is a subgroup of* Γ *of index 2.*

The orthogonal isomorphisms $V \to V$ *is a group under composition* $\circ$ *, the orthogonal group O(B).*

The mapping $\Psi : U \to S_U$ *is a multiplicative morphism from* Γ *into the orthogonal group O(B).*

NB: The form $x \to U\, x\, U^{-1}$ also gives orthogonal isomorphisms, but fewer and not with the same sort of uniqueness.

Lemma 6.8. The mapping $\Psi : U \to S_U$ *is a multiplicative morphism from* Γ *into the orthogonal group O(B).*

Proof: Follows from

$\Psi(T) \circ \Psi(U) = (-1)^{ks}\, T\, (-1)^{hs}\, U X U^{-1}\, T^{-1} = (-1)^{(h+k)s}\, (T U) X (T U)^{-1} = \Psi(T U)$.

Theorem 6.9. Assume $\mathbb{K}$ is a field not of characteristic 2, V has a finite basis and B is regular.
1. Then from the Cartan-Dieudonne theorem follows that any orthogonal isomorphism of V can be expressed as $S(x) = \hat{U}\, x\, U^{-1}$, where U is a h-versor with $h \le n = \dim(V)$.
2. $\forall_{x\in V}\ \hat{U}\, x\, U^{-1} = \hat{T}\, x\, T^{-1}$ imply $T \in \mathbb{K}^\times U$.
3. $\Psi : \Gamma \to O(B)$ is onto $O(B)$ with kernel $\Psi^{-1}(\mathrm{id}_V) = \mathbb{K}^\times$
Also Ψ maps Γ^+ onto $O^+(B)$, the orthogonal isomorphisms with determinant 1 called rotations.
4. Moreover, if $\forall_{x\in V}\ \psi(x) = \hat{T}\, x\, T^{-1} \in V$, then T is a versor.

Proof: 1. The Cartan-Dieudonne theorem states that an orthogonal isomorphism can be expressed as a composition of at most n reflections $\mathcal{R}^u$.
2. If $\hat{U}\, x\, U^{-1} = \hat{T}\, x\, T^{-1}$ and $A = U^{-1}\, T$, then $x\, A = \hat{A}\, x$, which gives $2\, x \rfloor A = 0$. Thus $A \in \mathbb{K}\backslash\{0\}$.
3. From (2) and last theorem.
4. Like the proof for Theorem 6.5 (2b), we get $\psi(x)\cdot\psi(y) = x\cdot y$, and $\psi \in O(B)$, as $\psi^{-1}(x) = T^{-1}\, x\, \hat{T}$.
Now from (1) follows $\psi = S_U$ for some U, and from (2) $T / U \in \mathbb{K}^\times$.

It is possible to remove some of the redundancy of the versor U without spoiling the group properties by using the mapping Φ below.
Corollary 6.10. The mapping $\Phi : U \to U\,\tilde{U} \in \mathbb{K}^\times$ is a multiplicative morphism, $\Phi(\Gamma)$ is a multiplicative group, and $\Phi(\Gamma) = \Phi(\Gamma)\,\mathbb{K}^{\times 2}$.
2. Assume $\mathbb{K}^\times = S \times (\mathbb{K}^\times)^2$ as direct product of multiplicative subgroups S and $(\mathbb{K}^\times)^2$, like e.g. $\mathbb{R}^\times = \{\pm 1\} \times (\mathbb{R}^\times)^2$ or $\mathbb{C}^\times = \{1\} \times (\mathbb{C}^\times)^2$.
Then each $U \in \Gamma$ can be normalized as $t\,U$, such that $\Phi(t\,U) \in S$, and t is unique apart from a factor ± 1.
Define $\mathrm{pin}(B) = \Phi^{-1}(S)$ and $\mathrm{spin}(B) = \mathrm{pin}(B) \cap \Gamma^+$, $\mathrm{pin}^+(B) = \Phi^{-1}(1)$ and $\mathrm{spin}^+(B) = \mathrm{pin}^+(B) \cap \Gamma^+$.
If $U \in \mathrm{pin}(B)$, then $S_U(x) = s^{-1}\,\hat{U}\, x\, \tilde{U}$, where $s = \Phi(U) \in S$

Proof: 1. $U\,\tilde{U} = \tilde{U}\,U = \langle U\,\tilde{U}\rangle$, and
$\Phi(U\,T) = \langle (U\,T)\,(U\,T)^\sim\rangle = \left\langle (U\,T)\left(\tilde{T}\,\tilde{U}\right)\right\rangle = \left\langle \tilde{U}\left(U\,T\,\tilde{T}\right)\right\rangle = \Phi(U)\,\Phi(T)$
If $U = u_1 \dots u_h$, then $U\,\tilde{U} = u_1^2 \dots u_h^2 \in \mathbb{K}^\times$ and is invertible, as each u_k is. Also $\Phi\left(U^{-1}\right) = \left(u_h^{-1} \dots u_1^{-1}\right)\left(u_1^{-1} \dots u_h^{-1}\right)^\sim = u_1^{-2} \dots u_h^{-2} \in \mathbb{K}^\times$, and $\Phi(U)\,\Phi\left(U^{-1}\right) = \Phi\left(U^{-1}\right)\Phi(U) = 1$. Thus $\Phi(\Gamma)$ is a group, as $s, t \in \Phi(\Gamma) \Rightarrow s^{-1}, s\,t \in \Phi(\Gamma)$.
If $v \in \mathbb{K}^\times$, then $U \in \Gamma \Leftrightarrow v\,U \in \Gamma$ and therefore $s^2\,\Phi(\Gamma) = \Phi(\Gamma)$
2. (a) Assume $\Phi(U) = s\,k^2$, $s \in S$, then $\Phi(t\,U) = s \Leftrightarrow s\,(t\,k)^2 = s \Leftrightarrow t = \pm k^{-1}$.
(b) Follows from the definition of S_U, and $U \in \mathrm{pin}(B) \Rightarrow U\,\tilde{U} = s \in S \Rightarrow U^{-1} = s^{-1}\,\tilde{U}$, $s \in S$.
NB: The concept covering requires topological spaces, which is not the case here in this general setup.

Example. In $\Lambda(V)$ any bijective linear transformation is an orthogonal isomorphism, but none has versor forms.
Example. In $\mathbb{R}^{2,1}$ let $u = (e_1 + e_2)\big/\sqrt{2}$, $f_1(x) = u\, e_2\, x\, e_2\, u$ and $f_2(x) = -u\, e_3\, x\, e_3\, u$. Then f_1 and f_2 are in $\mathrm{spin}(B)$.

Chapter 7 Finer structures in Clifford algebra

Some isomorphisms of Clifford algebras

Theorem 7.1. Let (a_i) be an orthogonal basis for V, $V_\kappa = V \oplus \mathbb{K}\, a_\kappa$, a_κ orthogonal to V, $a_\kappa^2 = \varepsilon$ invertible, and B_κ the extension of B to V_κ. Define a linear mapping $f : V \to C\ell(B_\kappa)^+$ by $u \to u\, a_\kappa$. Then f extends uniquely to an algebra isomorphism $F : C\ell(-\varepsilon B) \to C\ell(B_\kappa)^+$.

Proof: We may assume $(a_i \mid i \in M)$ is the standard basis for V, and in the index ordering makes κ last. From f linear and $f(u)^2 = u\, a_\kappa\, u\, a_\kappa = -\varepsilon\, u^2$ follows the extension of f by universality to $F : C\ell(-\varepsilon B) \to C\ell(B_\kappa)^+$ as algebra morphism.
Products $a_i\, a_\kappa$, $i \in M$ span $C\ell(B_\kappa)^+$ as algebra, as $a_i\, a_\kappa\, a_j\, a_\kappa = -\varepsilon\, a_i\, a_j = -\varepsilon\, a_{\{i,j\}}$, $i \neq j$, and any $a_H \in A^+$ is product of 2-grade basis elements.
F is bijective, as the basis $(a_K, K \text{ finite } K \subseteq M)$ is mapped bijectively onto the basis $(\pm \varepsilon^p\, a_K, K \text{ even or } \pm \varepsilon^p\, a_{K \cup \{\kappa\}}\ K \text{ odd} \mid K \subseteq M)$ for $C\ell(B_\kappa)^+$.

Example. In the case $\mathbb{R}_{p,q}$ we get we get $F : \mathbb{R}_{q,p} \to \mathbb{R}^+_{p+1,q}$ when $\varepsilon = 1$, and $F : \mathbb{R}_{p,q} \to \mathbb{R}^+_{p,q+1}$ when $\varepsilon = -1$.
Thus $\mathbb{R}_{q,p-1} \simeq \mathbb{R}^+_{p,q}$ when $p \geq 1$, and $\mathbb{R}_{p,q-1} \simeq \mathbb{R}^+_{p,q}$ when $q \geq 1$. Hence $\mathbb{R}_{q,p-1} \simeq \mathbb{R}_{p,q-1}$ when $p,\, q \geq 1$.
The algebra isomorphism is not as Clifford algebras.

Center. Simplicity.

*Definition 7.2. An algebra A is called simple, if A has no twosided ideals other than 0 and A.
The center $Z = Z(A)$ of an algebra A consists of the elements commuting with all the elements of A.*

*Theorem 7.3. Assume $\mathbb{K}$ is a field of characteristic$\neq$2. Then,
if $|M|$ is finite and odd, then $Z = Z(C\ell(B)) = \mathbb{K}\, e_M + \mathcal{G}(V_0)^+$, and otherwise $Z = \mathcal{G}(V_0)^+$,* where *V_0 is the radical or kernel of B.*

Proof: Clearly Z is a linear subspace of $C\ell(B)$.
We have $X \in Z \Leftrightarrow \forall_{Y \in C\ell(B)} : Y X - X Y = 0 \Leftrightarrow \forall_{Y \in C\ell(B)} : Y \hat{X} - \hat{X} Y = 0 \Leftrightarrow \hat{X} \in Z$.
Also from $X_0 = \langle X \rangle_{\text{even}} = \left(X + \hat{X}\right)/2$ and $X_1 = \langle X \rangle_{\text{odd}} = \left(X - \hat{X}\right)/2$ follows $X \in Z \Leftrightarrow X_0, X_1 \in Z$.
1. Assume $X \in Z$. As $\forall_i : 0 = (e_i X_1 - X_1 e_i)/2 = e_i \wedge X_1$, and using theorem 5.2 (11a) we get, if $|M|$ is finite and odd, that $X_1 \in \mathbb{K}\, e_M$,
and otherwise that $X_1 = 0$.

Furthermore from $\forall_i : 0 = (e_i X_0 - X_0 e_i)/2 = e_i \rfloor X_0$ follows $X_0 \in \mathcal{G}(V_0)^+$ using theorem 5.2 (11b).
2. Verification. From $X \in \mathcal{G}(V_0)^+$ follows $\forall_i : (e_i X - X e_i)/2 = e_i \rfloor X = 0$ and therefore $X \in Z$.
Finally $|M|$ is finite and odd implies $\forall_i : e_i e_M = e_M e_i$ and therefore that $e_M \in Z$.

Lemma 7.4. For algebra A and $f \in A$ assume f is idempotent, $f^2 = f$. Then
1. $(1-f)^2 = (1-f)$ *and* $f(1-f) = 0$.
2. $\mathcal{I}_- = A f$ *is a left ideal.*
3. $P_- : X \to X f$ *is an algebraic projection onto* $\mathcal{I}_-$*, such that*
(3bcd): $P_-^2 = P_-$, $(1-P_-)^2 = (1-P_-)$ *and* $P_-(1-P_-) = 0$, all with $1 = \mathrm{id}_V$.
Moreover $X \in \mathcal{I}_- \Rightarrow P_-(X) = X$
4. If $f \in Z(A)$, then $\mathcal{I}_-$ is as a twosided ideal, and $P_- : A \to \mathcal{I}_-$ is an algebra homomorphism.
As $(1-f)$ has the same properties as those mentioned of f, it give likewise rise to left ideal $\mathcal{I}_+ = A(1-f)$ and $P_+ = 1 - P_-$.
Analogous statements to (1-3) holds and furthermore
5. $\mathcal{I}_- \oplus \mathcal{I}_+ = A$

Proof: 1. Obvious
2. $A\,\mathcal{I}_- = A^2 f = A f = \mathcal{I}_-$
3. (3b) $P_-^2(X) = X f^2 = X f = P_-(X)$ and this imply (3c,d)
(3e) $X \in \mathcal{I}_- \Rightarrow X = Y f \Rightarrow P_-(X) = Y f^2 = Y f = X$
(3a) $P_-(X) = X f \in \mathcal{I}_-$ and (3e) imply P_- is onto $\mathcal{I}_-$
4. $P_-(X)\,P_-(Y) = X f Y f = X Y f^2 = X Y f = P_-(X Y)$.
5. $\mathcal{I}_- + \mathcal{I}_+ \supseteq (P_- + P_+)(A) = A$ and by (3e,d) $X \in \mathcal{I}_- \cap \mathcal{I}_+ \Rightarrow X = P_+(X) = P_-(P_+(X)) = 0$

Theorem 7.5. Assume $\mathbb{K}$ is a field of characteristic≠2 and B is regular. Then
1. If $|M|$ is even or infinite, then $C\ell_V(B)$ is simple.
2. Assume $|M|$ finite and odd. Then
$\mathcal{I}$ is a non-trivial ideal $\Leftrightarrow \exists_\lambda : \mathcal{I} = C\ell(B)(1 + \lambda e_M)$ *and* $\lambda^2 e_M^2 = 1$.
3. Assume $|M|$ finite and odd, and $\lambda^2 e_M^2 = 1$. Then
(3a) $f_\pm = (1 \pm \lambda e_M)/2 \in Z(C\ell(B))$, $f_\pm f_\mp = 0$ *and* $f_\pm^2 = f_\pm$.
This gives projections and algebra homomorphisms $P_\pm(X) = X f_\pm$ onto proper ideals $\mathcal{I}_\pm = P_\pm(C\ell(B))$, such that
$P_- + P_+ = \mathrm{Id}_{C\ell(B)}$, $\quad P_- P_+ = 0$, $P_\pm^2 = P_\pm$, *and* $\mathcal{I}_- \oplus \mathcal{I}_+ = C\ell(B)$.
(3b) $P_\pm(X) \cdot Y = X \cdot P_\pm(Y)$
(3d) $C\ell(B)^+$ isomorphic to each ideal $\mathcal{I}_\pm$ by the restriction of $P_\pm$ to $C\ell(B)^+$.
(3e) $C\ell(B)^+$ and $\mathcal{I}_\pm$ are all simple.
(3f) The only non-trivial ideals in $C\ell_V(B)$ are $\mathcal{I}_-$ and $\mathcal{I}_+$.

Proof: Assume $X = \Sigma_{K\in\mathcal{E}} \lambda_K e_K \neq 0,\ \lambda_K \neq 0$ belongs to a non-trivial ideal $\mathcal{I}$.
Also assume X is chosen, such that the expansion has a minimal number of nonzero coefficients.
As each e_K is invertible, then after division with one of them we may assume $\emptyset \in \mathcal{E}$ and $\lambda_\emptyset = 1$.
Let $\theta(X)$ be the proposition: X has an even term, $\lambda_H e_H$ with a factor $e_i,\ i \in H$, or an odd term, $\lambda_H e_H$ with a factor $e_i,\ i \in M\backslash H$.
If $\theta(X)$, then $X = 1 + \lambda_H e_H + X_{\text{remaining}} \in I$ and get $e_i X / e_i = 1 - \lambda_H e_H + e_i X_{\text{remaining}} / e_i \in I$.
Therefore $X + e_i X / e_i$ simplified has fewer terms than X, as the terms $\lambda_H e_H$ cancel and each $e_i \lambda_K e_K / e_i = \pm\lambda_K e_K$.
This contradiction shows the negation of $\theta(X)$ holds.

1 (Case $|M|$ even or infinite). As H odd $\Rightarrow M\backslash H \neq \emptyset$, X can only have even terms, and here only 1. Thus $\mathcal{I}$ is trivial and $C\ell_V(B)$ is simple.
2 (Case $|M|$ odd). Here $e_M \in Z$. Let $\lambda = \lambda_M$. As $M = H$ is possible, $X = 1 + \lambda e_M$. Now $(1 - \lambda e_M) X = 1 - \lambda^2 e_M^2 \in \mathcal{I} \bigcap \mathbb{K} = \{0\}$, as the ideal is non-trivial, and $e_M^2 = \lambda^{-2}$ is necessary for this, and by (3f) below sufficient.
3. (3a) Follows from Lemma 7.4 follows, as $f_+ = 1 - f_-$
(3b) $P_\pm$ is symmetric, as $P_\pm(X) \cdot Y = \langle X(1 \pm \lambda e_M) Y\rangle / 2 = X \cdot P_\pm(Y)$.
(3d) The restriction of P_+ to $C\ell(B)^+$ is injective, as the odd part of $P_+(X) = X f_+$ is $X/2$.
If $|K|$ is odd, then, as $P_+ P_- = 0$, $P_+(e_K) = P_+(P_+(e_K) - P_-(e_K)) = P_+(\lambda e_K e_M)$.
Thus $P_+(e_K) = \lambda\, \sigma(K, M)\, P_+(e_{M\backslash K})$, where $M\backslash K$ is even and $\lambda\, \sigma(K, M) \neq 0$.
Hence $P_+(C\ell(B)^+) = \text{span}\{P_+(e_K) \,|\, K \in \mathcal{F},\ K \text{ is even}\} = \text{span}\{P_+(e_K) \,|\, K \in \mathcal{F}\} = \mathcal{I}_+$.
Likewise can be proved that the restriction of P_- to $C\ell(B)^+$ is an algebra isomorphism.
(3e) $C\ell(B)^+$ is simple follows from theorem 7.1 by the isomorphism $C\ell(-\varepsilon B) \to C\ell(B_\kappa)^+$, as the basis for $C\ell(-\varepsilon B)$ has even size $|M - 1|$.
(3f) Let $\mathcal{I}$ be a non-trivial ideal such that $f_+ = (1 + \lambda e_M) \in \mathcal{I}$.
Hence $f_+ \in \mathcal{I} \bigcap \mathcal{I}_+$ implies $\mathcal{I}_+ \subseteq \mathcal{I}$. As $\mathcal{I} \bigcap \mathcal{I}_-$ is an ideal and $\mathcal{I}_-$ is simple, we must have $\mathcal{I} \bigcap \mathcal{I}_- = \{0\}$.
That $\mathcal{I} \subseteq \mathcal{I}_+$ follows from $P_-(I) = I\,(1 - \lambda e_M)/2 \subseteq \mathcal{I} \bigcap \mathcal{I}_- = \{0\}$ and $X \in I \Rightarrow X = P_-(X) + P_+(X) = P_+(X) \in \mathcal{I}_+$. Thus $\mathcal{I} = \mathcal{I}_+$.

Theorem 7.6. Assume B is regular and $\mathbb{K}$ *is a field of characteristic*$\neq 2$, *that* $|M|$ *is finite* >1 *and odd, and also that* $\lambda \in \mathbb{K}$ *can be found, such that* $\lambda^2 e_M^2 = 1$. *Then*
1. In $\mathcal{A}(V, B)$ *exists besides* $C\ell_V(B)$ *only the algebras* $U_\pm = C\ell\,(B) / \mathcal{I}_\pm$.
2. If $\tilde{e_M} = e_M$, *then* $C\ell(B) / \mathcal{I}_\pm$ *has reversion and no conjugation.*
Otherwise, if $\overline{e_M} = e_M$, *then* $C\ell(B) / \mathcal{I}_\pm$ *has conjugation and no reversion.*

Proof: Set $f_\pm = (1 \pm \lambda\, e_M)/2$.
1. By theorem 7.4 $C\ell_V(B)$ has proper ideals $\mathcal{I}_\pm$ and only these, and by theorem 3.5, it remains to show properties 1-3 in definition 1.1 for $U_\pm$.
The natural mapping $\phi_\pm : C\ell(B) \to U_\pm$ has kernel $\mathcal{I}_\pm$.
Assume e.g. $k + x \in \mathcal{I}_- \bigcap (\mathbb{K} \oplus V)$. Then $k + x = Y f_-$ giving $(k + x) f_+ = 0$ or $k + x + \lambda\, k\, e_M + \lambda\, x\, e_M = 0$. As $|M| > 1$, these four elements has different grades, if not equal to zero.
Likewise for $k + x \in \mathcal{I}_+ \bigcap (\mathbb{K} \oplus V)$.
Hence $k = x = 0$, and $\phi_\pm$ is injective on $\mathbb{K} \oplus V$, which can be identified with its image by $\phi_\pm$. This gives property 3 and 1, 2 are now obvious.
2. If $\tilde{e_M} = e_M$, then $\tilde{f_\pm} = f_\pm$ and $\mathcal{I}_\pm = (\mathcal{I}_\pm)^\sim$, whence $C\ell(B)/\mathcal{I}_\pm$ may have reversion transferred.
Likewise, if $\tilde{e_M} = -e_M$, i.e. $\overline{e_M} = e_M$, then $C\ell(B)/\mathcal{I}_\pm$ may have conjugation transferred.
As by theorem 5.6 a main automorphism does not exists in $C\ell(B)/\mathcal{I}_\pm$, only one of the transformations reversion and conjugation can exist.

Example. In $\mathbb{R}_{1,0,1}$ bilinearform B is not regular, has diagonal matrix $(1,\ 0)$ and $\mathcal{I} = e_2\, \mathbb{R}_{1,0,1}$ is the ideal span$(e_2,\ e_{12})$.
The algebra $\mathbb{R}_{1,0,1} \big/ \mathcal{I} \simeq \mathbb{R}_1$ is spanned by $\{1,\ e_1\}$, which has a main automorphism. Thus regularity is essential in theorem 5.6.

Example. $\mathbb{R}_{p,q}$. If $p - q \equiv 2\,h + 1 \pmod 4$, then $e_M^2 = (-1)^h$.
Hence if $p + q$ is infinite, even or $p - q \equiv 3 \pmod 4$, then $\mathbb{R}_{p,q}$ is simple.
Otherwise $p - q \equiv 1 \pmod 4$, and $\mathbb{R}_{p,q} = \mathcal{I}_- \oplus \mathcal{I}_+$.
If furthermore q is odd/even, then $\mathbb{R}_{p,q} \big/ \mathcal{I}_\pm$ has conjugation/reversion according to the parity of q.
This follows by interger calculation, as $q = 2\,r - \delta$ and $p = q + 1 + 4\,s$ imply that the reversion exponent $(p + q)\,(p + q - 1)/2 \equiv \delta \pmod 2$.

Example. $\mathbb{C}_p$. If p is even or infinite $\mathbb{C}_p$ is simple.
Otherwise, as $e_M^2 = \lambda^{-2}$ can be solved for λ, $\mathbb{C}_p$ has ideals $\mathcal{I}_\pm$.
If $p \equiv 1 \pmod 4$, then $\mathbb{C}_p \big/ \mathcal{I}_\pm$ has a reversion.
If $p \equiv 3 \pmod 4$, then $\mathbb{C}_p \big/ \mathcal{I}_\pm$ has a conjugation.
Example. $\mathbb{C}_3$. As $\overline{e_M} = e_M$ and $\mathcal{I}_\pm = \mathbb{C}_3\,(1 \pm e_M)/2 = \overline{\mathcal{I}}_\pm$, $\mathbb{C}_3 \big/ \mathcal{I}_\pm$ has a conjugation transferred from $\mathbb{C}_3$.

Linear independency Clifford products. The quantization transformation

Lemma 7.7. Let $x_i \in V$, *then* $x_1\, x_2 \ldots x_p - x_1 \wedge x_2 \wedge \ldots \wedge x_p \in \Lambda_{<p}(V)$.

Proof: Induction after p is used. It is trivial for $p = 0, 1$. Assume the statement is true for $1 < p < r$. If $X = x_2 \ldots x_r$ and $Y = x_2 \wedge \ldots \wedge x_r$, then $x_1\, X - x_1 \wedge Y = x_1 \wedge (X - Y) + x_1 \rfloor (X - Y) + x_1 \rfloor Y \in \Lambda_{<r}$.
The assertion follows now from the induction principle.

Theorem 7.8. Let $(x_i \mid i \in I)$ *be linear independent in* V.

1. Then $(x_K \mid K \subseteq I, K$ *finite*$)$ *is linear independent.*
2. Assume $(x_i \mid i \in I)$ *is a basis for* V, *and* $K \subseteq I$, K *finite.*
Then the quantization transformation $f : \mathcal{G}(V) \to \mathcal{G}(V)$ *is well-defined by linearity and* $f(x_{\wedge K}) = x_K$.
Moreover f *is bijective and* (x_K) *is a basis for* $\mathcal{G}(V)$.

Proof: 1. Assume $Y = \Sigma_{K\in\mathcal{E}} \lambda_K x_K$ with all $\lambda_K \neq 0$ and $\mathcal{E} \neq \emptyset$, and set $X = \Sigma_{K\in\mathcal{E}} \lambda_K x_{\wedge K}$ and $r = \max \{|K| \mid K \in \mathcal{E}\}$.
According to lemma 7.7, $X - Y = \Sigma_{K\in\mathcal{E}} \lambda_K(x_{\wedge K} - x_K) \in \Lambda_{<r}(V)$. If $Y = 0$, this gives a contradiction, and the assertion follows.
2. As $(x_{\wedge K})$ is a basis for $\mathcal{G}(V)$, f is well-defined and by the proof of (1) injective.
If f is not surjective, select if possible $X \in \mathcal{G}(V)\backslash f(\mathcal{G}(V))$ with lowest grade of highest grade term. Then $X \neq 0$.
If $X - f(X) \in f(\mathcal{G}(V))$ then $X \in f(\mathcal{G}(V))$, which is a contradiction. Hence $X - f(X) \in \mathcal{G}(V)\backslash f(\mathcal{G}(V))$, and has according to lemma 7.7 lower highest grade than X, which gives a contradiction. Thus f is bijective and maps a basis onto a basis.

Parity. Twisted algebra

Definition 7.9. In $C\ell(B)$ *define parity of* X *by* $\text{par}(X) = p \iff \text{grade}(X) \equiv p \pmod 2$.
Also set $C\ell(B)^- = \{X \mid \text{par}(X) = 1\}$ *and* $C\ell(B)^+ = \{X \mid \text{par}(X) = 0\}$
Parity makes $C\ell(B)$ *a graded algebra:* $\text{par}(X) = r$ and $\text{par}(Y) = s \Rightarrow \text{par}(X\,Y) = r + s \pmod 2$.

Proof: Follows from theorem 2.2, as factor reductions for products are even in number.

With a new product, $X \,\tau\, Y$ in $C\ell(B, V)$ we get an algebra $C\ell(B, V)^{\text{tw}}$ isomorphic to $C\ell(-B, V)$.
Definition 7.10. To every Clifford algebra $C\ell(B, V)$ *is associated a twisted algebra* $C\ell(B, V)^{\text{tw}}$ *in the same linear space with multiplication defined by linearity and* $X \,\tau\, Y = (-1)^{rs} X\,Y$ *when* $\text{par}(X) = r$ *and* $\text{par}(Y) = s$
1. This gives an algebra structure, such that $x \,\tau\, x = -x^2$ *for* $x \in V$. *The twisted of the twisted algebra is the original.*
2. The universal extension of id_V *is an algebra isomorphism* $F : C\ell(-B, V) \to C\ell(B, V)^{\text{tw}}$.

Proof: Let $\text{par}(X) = r$, $\text{par}(Y) = s$ and $\text{par}(Z) = h$.
1. Most are obvious. As associative law is multilinear, it needs only be verified for homogeneous elements. We get $(X \,\tau\, Y) \,\tau\, Z = (-1)^{(r+s)h} ((-1)^{rs} X\,Y) Z = (-1)^{rs+rh+sh} X\,Y\,Z$, and $X \,\tau (Y \,\tau\, Z) = (-1)^{r(s+h)} X \left((-1)^{sh}\, Y Z\right) = (-1)^{rh+rh+sh} X\,Y\,Z$.
2. As $F(e_K) = \pm e_K$, F maps a basis onto a basis bijectively, and therefore is bijective. Thus F is an

isomorphism.

Chapter 8 Chevalley's construction of Clifford algebras from tensor algebras

Chevally's construction of Clifford algebras is based on tensor algebra and do not require all the properties in definition 1.1.
He start with a quadratic form, but he proves in a short note that it is equivalent to use any bilinear form possible non-symmetric [2, p.76 I.2.2].
In this chapter B is an arbitrary bilinear form on V.

This universality statement is a key point:
Theorem 8.1. Let $\mathcal{T} = \mathcal{T}(V, \otimes)$ be the tensor algebra over V. For any algebra A over $\mathbb{K}$ and any linear mapping $\tau : V \to A$, there is a unique algebra morphism $\mathrm{T} : \mathcal{T} \to A$ that extends τ.

Definition 8.2. Let $\mathcal{I} = \mathcal{I}(V, B)$ be the two-sided ideal in $\mathcal{T} = \mathcal{T}(V)$ generated by $\mathcal{S} = \{x \otimes x - B(x, x)\, 1_{\mathcal{T}} \mid x \in V\}$.
The Clifford algebra $CC\ell_V(B)$ is then defined as the quotient algebra $CC\ell = \mathcal{T} / \mathcal{I}$ together with $\hat{\pi} : \mathcal{T} \to CC\ell$, the canonical algebra morphism.

Example. Let $\mathbb{K} = \mathbb{Z}_6$, $V = \mathbb{Z}_3 \times \mathbb{Z}_6$, $(e_1,\ e_2)$ the standard basis and $B = \mathrm{diag}(1,\ 1)$. This imply $3\, e_1 = 6\, e_2 = 0$.
Moreover $\hat{\pi}(3) = 3\ 1_{CC\ell} = 3\, \hat{\pi}\left(e_1^2\right) = 3\, \hat{\pi}(e_1)\, \hat{\pi}(e_1) = \hat{\pi}(3\, e_1)\, \hat{\pi}(e_1) = 0$ and $\hat{\pi}(3\, e_2) = \hat{\pi}(3)\, \hat{\pi}(e_2) = 0$, but $3 \neq 0$ in $\mathbb{K}$ and $3\, e_2 \neq 0$ in V.
Thus $\hat{\pi}$ is neither injective on $\mathbb{K}$ or on V.

This simple approach gives the problem that $\hat{\pi}$ need not to be injective on $\mathbb{K}$ or on V, implying $\mathbb{K}$ and V can not generally be identified with their images by $\hat{\pi}$. Thus the universality principle in definition 1.1 is unusable. However the Chevally's construction is in harmony with a universality principle, not for the algebras, but for the morphisms $\hat{\pi} : \mathcal{T} \to CC\ell$.

Definition 8.3. Let $\mathcal{K}(V, B)$ be the category of linear mappings f from V into an algebra A, such that $f(x)^2 = B(x, x)\, 1_A$.
A mapping $\omega : V \to U$ in $\mathcal{K}(V, B)$ is said to be universal, if for every linear mapping $f : V \to A$ in $\mathcal{K}(V, B)$, there is a unique algebra morphism $F : U \to A$ such that $F \circ \omega = f$. (i.e. $f : V \xrightarrow{\omega} U \xrightarrow{F} A$)

As we shall show a Cliford algebra in the version presented in definition 1.1 is also a Chevalley Cliford algebra
Theorem 8.4.
1. $\pi = \hat{\pi}\,|_V$ from definition 8.2 is a universal object in $\mathcal{K}(V, B)$.
2. Assume B is symmetric.
Then $\pi : V \to CC\ell$ is injective and has an extension $G : C\ell_V(B) \to CC\ell$ to an algebra isomorphism, and therefore $G(1_{C\ell_V(B)}) = 1_{CC\ell}$.

Proof:

$$\begin{array}{ccccc} V & \subset & \mathcal{T} & \overset{\hat{\pi}}{\longrightarrow} & \mathcal{T}\big/\mathcal{I} = CC\ell \\ id\downarrow & & \mathrm{T}\downarrow & & F\downarrow \\ V & \overset{f}{\longrightarrow} & A & = & A \end{array}$$

1. π is an object in $\mathcal{K}(V, B)$, as $\pi(x) = x \otimes x + \mathcal{I} = B(x, x)\, 1_{\mathcal{T}} + \mathcal{I} = B(x, x)\, 1_{CC\ell}$.
If $f : V \to A$ is in $\mathcal{K}(V, B)$, then by tensor universality there is an unique algebra morphism $\mathrm{T} : \mathcal{T} \to A$ that extends f.
As $\mathrm{T}(x \otimes x) = \mathrm{T}(x)^2 = f(x)^2 = B(x, x)\, 1_A = \mathrm{T}(B(x, x)\, 1_{\mathcal{T}})$ implies $\mathcal{S} \subseteq \mathrm{T}^{-1}(0)$ and therefore $\mathcal{I} \subseteq \mathrm{T}^{-1}(0)$, there exists an algebra morphism $F : CC\ell \to A$, such that $\mathrm{T} = F \circ \hat{\pi}$. Thus $F \circ \pi = f$.
If $G : CC\ell \to A$ is an algebra morphism, such that $f = G \circ \pi$, then $G \circ \hat{\pi}$ is an algebra morphism $\mathcal{T} \to A$ that extends f.
Therefore by theorem 8.1 $\mathrm{T} = G \circ \hat{\pi}$. As $\hat{\pi}$ is onto $CC\ell$, we get $G = F$ showing F is unique.
2. Assume B is symmetric.
From (1) follows, that to $\mathrm{id}_V : V \to C\ell_V(B)$ there exists an unique algebra morphism $F : CC\ell \to C\ell_V(B)$, such that $F \circ \pi = \mathrm{id}_V$. Thus π is injective.
From universality of $C\ell_V(B)$ follows, that to $\pi : V \to CC\ell$ there exists an unique algebra morphism $G : C\ell_V(B) \to CC\ell$, such that $G\,|_V = \pi$.
As $F \circ G : C\ell_V(B) \to C\ell_V(B)$ is an algebra morphism that on V is the identity, then universality of $C\ell_V(B)$ implies $F \circ G = \mathrm{id}_{C\ell_V(B)}$
Now $G \circ F : CC\ell \to CC\ell$ is an algebra morphism for which $G \circ F \circ \pi = G \circ \mathrm{id}_V = \pi$. Then by universality of $CC\ell$ from (1) there is a unique algebra morphism $H : CC\ell \to CC\ell$ such that $H \circ \pi = \pi$, which of course is $\mathrm{id}_{CC\ell}$. Thus $G \circ F = \mathrm{id}_{CC\ell}$.
Hence $G : C\ell_V(B) \to CC\ell$ is an isomorphism and both $G\,|_V = \pi$ and $G\,|_{\mathbb{K}}$ are injective

Even for symmetric bilinearforms Chevalley Cliford algebras are more general that the version presented in definition 1.1.
Example. Assume B is regular and $\mathbb{K}$ is a field of characteristic$\neq 2$, $M = \{1\}$, and $e_1^2 = 1$.
As $\dim(C\ell\,(B)\big/\mathcal{I}_\pm) = 1$, $\mathbb{K} \oplus V$ can not be mapped injectively into $C\ell(B)\big/\mathcal{I}_\pm$. Thus these algebras does not comply with definition 1.1.
However $f : V \to C\ell(B)\big/\mathcal{I}_\pm$, the quotient mapping of id_V, belongs to $\mathcal{K}(V,\ B)$, as $f(x)^2 = x^2 + \mathcal{I}_\pm\ = B(x,\ x) + \mathcal{I}_\pm$.
Hence $C\ell(B)\big/\mathcal{I}_\pm$ Chevalley Cliford algebras, and the only besides $C\ell(B)$ and the null-algebra in $\mathcal{K}(V,\ B)$.

Conclusion

On elementary basis the different algebras are compactly constructed.
An extensive formula collection is proved.
The universality principle is described, and its forcefulness demonstrated in many ways:

To fully define Clifford algebras.
To prove in-dependency of orthogonal basis**.**
To define the main automorphism and the reversion.
In various proofs.
To establish connection to Chevalley's tensor based Clifford algebras construction.
Non-universal Clifford algebra's.

Several types of projections are treated among which is parallel projection.
A comprehensive formula collection is established.
Precise conditions for simplicity are found.
Various conditions for non-universality is found as well as connections to the main automorphism, the reversion, and the Clifford conjugation.

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