Continued Fractions and Semiconvergents as approximations to rational numbers

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Abstract

There are several sources expanding the approximation theme and not so many about semiconvergents, and they are mostly without proofs. This note with proofs in turbo style is intended to cover a part of the subject.

Introduction

I may be well known that the convergents of a continued fraction representation of a real number gives astonishing good rational approximations; but others in a certain way best approximations exists. E. g. some convergents of π are 3, $22/7$, $355/113$; but in between are other good approximations like $201/64$, $311/99$. There are several sources expanding the approximation theme and not so many about semiconvergents, and they are mostly without proofs. This note with proofs in turbo style is intended to cover a part of the subject.

Example. Let us look at $x = \exp(1/9) = 1.1175190687418636486220597164816527772611027132027551\ldots$

with continued fraction found by calculating $a_0 = \lfloor x \rfloor = 1$,

$y_1 = 1/(x - a_0) = 8.5092573546217307631832728\ldots$, $a_1 = \lfloor y_1 \rfloor = 8$

$y_2 = 1/(y_1 - a_1) = 1.9636437076942875114478 93\ldots$, $a_2 = \lfloor y_2 \rfloor = 1$

$y_3 = 1/(y_2 - a_2) = 1.0377279403325345959495 54\ldots$, $a_3 = \lfloor y_3 \rfloor = 1$

$y_4 = 1/(y_3 - a_3) = 26.505555065714851004423 67\ldots$, $a_4 = \lfloor y_4 \rfloor = 26$, etc.

Then this is collected in the continued fraction notation

$x = [1,\ 8,\ 1,\ 1,\ 26,\ 1,\ 1,\ 44,\ 1,\ 1,\ 62,\ 1,\ 1,\ 80,\ 1,\ 1,\ 98,\ 1,\ 1,\ 116,\ 1,\ 1,\ 134,\ 1,\ 1,\ 152,\ 1,\ 1,\ 170,\ \cdots]$

which means $x = 1 + 1/(8 + 1/(1 + 1/(1 + 1/(26 + \cdots))))$

Now the partial continued fractions gives

$[1] = 1$,

$[1,\ 8] = 9/8$,

$[1,\ 8,\ 1] = 10/9$,

$[1,\ 8,\ 1,\ 1] = 19/17$,

$[1,\ 8,\ 1,\ 1,\ 26] = 504/451$, etc.

which are best approximations of form p/q to $\exp(1/5)$ of the so called second kind, where the measure of distance is $|p - x\,q|$. In the case $19/17$ this means that any other fraction r/s with $1 \le s \le 17$ has greater value of $|r - x\,s|$, than $|19 - x \cdot 17|$.

The common measure of distance for approximations is of course $|x - p/\,q|$ and determines best approximations of first kind, which turns out to include the best approximations of second kind.

In the our example the first best approximations of the first kind fractions are

$1 = [1]$,

$6/5 = [1,\ 5]$

$7/6 = [1,\ 6]$

$8/7 = [1,\ 7]$

$9/8 = [1,\ 8]$,

$10/9 = [1,\ 8,\ 1]$,

$19/17 = [1,\ 8,\ 1,\ 1]$,

$257/230 = [1,\ 8,\ 1,\ 1,\ 13]$

$276/247 = [1,\ 8,\ 1,\ 1,\ 14]$ and consecutively for the last number up to

$504/451 = [1,\ 8,\ 1,\ 1,\ 26]$, etc.

These fractions have strictly decreasing values of $|x - p/\,q|$. Also for any of these fractions p/q any other fraction r/s with $1 \le s \le q$

has greater value of $|r/s - x|$, than $|p/q - x|$.

Please observe $5/4 = [1, 4]$ is not included, as $|5/4 - \exp(1/9)| > |1 - \exp(1/9)|$; but $[1, 8, 1, 1, 13]$ is included.

Fundamental Facts about Continued Fractions

We have sometimes to distinguish between a continued fraction and its value, since the last not always determine the first.

Theorem. Define a simple finite continued fraction of order $N \geq 0$, as $x = [a_0, a_1, ..., a_N]$ with $a_i \geq 1$ for $i > 0$, its value $\langle x\rangle = a_0 + 1/(a_1 + \ldots + 1/a_N)$ or just x, and its n'th convergent $A_n = \langle[a_0, a_1, ..., a_n]\rangle$. Then

1. $A_N = [a_0, a_1, ..., [a_r, ..., a_N]]$
2. A_N is strictly decreasing in a_{2i+1} and strictly increasing in a_{2i}, if present.

3a. $A_0 < A_2 < \ldots < A_{2i} < \ldots \leq A_{2i+1} < A_{2i-1} < \ldots < A_1$

3b. $(-1)^n A_n < (-1)^n x \leq (-1)^n A_{n+1}$

4. $A_n = p_n/q_n$ for $n = 0, 1, \ldots, N$ with p_n, q_n $n \geq -2$ defined by

$\begin{pmatrix} p_{-2} \\ q_{-2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \ldots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Thus $\begin{pmatrix} p_{-1} \\ q_{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} \begin{pmatrix} a_n \\ 1 \end{pmatrix}$.

5a. $p_{n+1} q_n - p_n q_{n+1} = (-1)^n$

5b. $A_{n+1} - A_n = (-1)^n/(q_n q_{n+1})$ and $|x - A_n| \leq 1/(q_n q_{n+1})$

6. $A_N = A_0 + \Sigma_{i=0}^{N-1} (A_{i+1} - A_i) = a_0 + \Sigma_{i=0}^{N-1} (-1)^i/(q_i q_{i+1})$

7. $\gcd(q_n, q_{n-1}) = \gcd(p_n, p_{n-1}) = \gcd(p_n, q_n) = 1$

8. Let $n \geq 0$. Then $q_n = a_n q_{n-1} + q_{n-2}$ and $q_{-2} = 1, q_{-1} = 0, q_0 = 1$.
Furthermore $q_n \geq q_{n-1}$ and $q_n \geq F_{n+1}$, where
the Fibbonnachi sequence (F_n) is defined by $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$.
Let $\sigma = \left(1 + \sqrt{5}\right)\Big/2 = 1.618$ ... Then $F_n = (\sigma^n - (-\sigma)^{-n})\Big/\sqrt{5}$ and $q_n \geq \left\lfloor \sigma^{n+1}\Big/\sqrt{5} + 0.3 \right\rfloor$

9. Suppose $[a_0, a_1, ..., a_n]$ is a simple finite continued fraction for all $n \geq 0$. Then $x = [a_0, a_1, ..., a_n, ...]$ is called a simple infinite continued fraction with order $N = \infty$ and, if $A_n \to x$ for $n \to \infty$, value $\langle x\rangle$ or just x.
This limit exists, and $x = a_0 + \Sigma_{i=0}^{\infty} (-1)^i/(q_i q_{i+1}) = [a_0, a_1, ..., [a_r, ..., a_n, \ldots]]$

10. Define $x_n = [a_n, a_{n+1}, \ldots]$. Then $x = [a_0, a_1, ..., a_{n-1}, \langle x_n\rangle]$

11. If $N > n$ then $(-1)^n (x - A_n) = \frac{1}{q_n (q_n x_{n+1} + q_{n-1})}$ and

$$q_n^{-2}/2 \leq 1/(q_n (q_n + q_{n-1})) \leq \left|x - A_n\right| \leq 1/(q_n q_{n+1}) \leq q_n^{-2}$$

Proof: 1-2. (1) is obvious is and (2) follow by induction, as $A_N = [a_0, [a_1, ..., a_N]]$.

3. By (1,2) we get $[a_n] < [a_n, a_{n+1}] \Rightarrow A_n < x \leq A_n \Rightarrow$ (3 b), $[a_{2i-2}] < [a_{2i-2}, a_{2i-1}, a_{2i}] \Rightarrow A_{2i-2} < A_{2i}$ and
$[a_{2i-1}] < [a_{2i-1}, a_{2i}, a_{2i+1}] \Rightarrow A_{2i+1} < A_{2i-1}$

4-6. Let $[a_k, ..., a_N] = r_k/s_k$. As $[a_k, ..., a_N] = a_k + 1/[a_{k+1}, ..., a_N] \Rightarrow r_k/s_k = (a_k r_{k+1} + s_{k+1})/r_{k+1}$
we get $\begin{pmatrix} r_k \\ s_k \end{pmatrix} \sim \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r_{k+1} \\ s_{k+1} \end{pmatrix}$ and thus $\begin{pmatrix} r_0 \\ s_0 \end{pmatrix} \sim \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \ldots \left(\begin{pmatrix} a_N & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$.

Now the trick is to define p_n, q_n as stated in (4) implying $A_n = p_n/q_n$,
$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \ldots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (*), and $\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$.

By (*) $\det\left[\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix}\right] = (-1)^n$, and dividing this with $q_n q_{n+1}$ gives the next formula, from which the remaining is obvious.

7. By (5a).

8. By (4) we have $q_n = a_n q_{n-1} + q_{n-2}$ and the first values. By induction is easily shown for $n \geq 0$

(i) $q_n \geq 1$, (ii) $q_n \geq q_{n-1}$, (iii) $q_{n+1} \geq q_n + q_{n-1}$, (iv) $q_n \geq F_{n+1}$.

The recurrence relation for F_n has characteristic polynomial $z^2 - z - 1$ with roots σ, $-\sigma^{-1}$ and the stated solution.

9. In the alternating series $\lim_{n\to\infty} 1/(q_n\, q_{n+1}) = 0$ by (8), and (6) implies the limit statement. Finally (1) and continuity of $z \to [a_0, a_1, ..., z]$ gives the remaining.

10. By (1, 9, 4)

11. As $(-1)^n (x - A_n) = (-1)^n \left(\frac{p_n\, x_{n+1} + p_{n-1}}{q_n\, x_{n+1} + q_{n-1}} - \frac{p_n}{q_n}\right) = \frac{1}{q_n\,(q_n\, x_{n+1} + q_{n-1})}$

and $\frac{1}{q_n\,(q_n + q_{n-1})} \leq \frac{1}{q_n\,(q_n\, x_{n+1} + q_{n-1})} \leq \frac{1}{q_n\, q_{n+1}}$

Corollary. For simple continued fractions $[a_0, a_1, ..., a_n, \ldots] = [b_0, b_1, ..., b_n, \ldots]$ of orders N and $M \geq N$ respectively holds

1. If the continued fractions are not identical, then $(\infty > M = N + 1) \wedge (a_N = b_{M-1} + 1) \wedge (b_M = 1)$ and $a_n = b_n$ for $n < N$.

Now for finite continued fractions let normal form be characterized by $(N = 0) \vee (a_N > 1)$ and long form by the opposite condition $(N > 0) \wedge (a_N = 1)$. Let also normal form cover infinite simple integer continued fractions. Then

2. A finite simple integer continued fraction represents a rational number, which can be represented in precisely two different ways. A normal form, that can be cast into long form by $[a_0, ..., a_N] \to [a_0, ..., a_N - 1, 1]$ and converse.

Proof: Assume the continued fractions are not identical. Let $m = \min\{k \mid a_k \neq b_k\}$, and set $y_k = [b_k, b_{k+1}, \ldots]$.

By theorem 1 (10, 2) $x_m = y_m$. Now three cases remain.

(i) $N, M \geq m + 1$. As $[a_m, x_{m+1}] = [b_m, y_{m+1}]$ we have a contradiction by

$a_m - b_m = 1/y_{m+1} - 1/x_{m+1} \in (-1, 1) \Rightarrow a_m = b_m$.

(ii) $N = M = m$ implying $a_m = b_m$, a contradiction.

(iii) $N = m \wedge M = m + 1$. We get $a_m = [b_m, b_{m+1}] \Leftrightarrow a_m = b_m + 1/b_{m+1} \Leftrightarrow a_m = b_m + 1 \wedge b_{m+1} = 1$.

Thus $(M > 0) \wedge (b_M = 1)$, while $N > 0 \Rightarrow a_N > 1$.

The remaining is obvious.

Corollary. Assume x is in normal form, $0 \leq n < N$ and $\Delta_k = x\, q_k - p_k = (x - A_k)\, q_k$. Then

1. $0 \leq (-1)^{n+1} \Delta_{n+1} < (-1)^n \Delta_n$
2. $0 \leq (-1)^n (A_{n+1} - x) < (-1)^n (x - A_n)$

Proof: As all is evident, if $x = A_{n+1}$, assume $x \neq A_{n+1}$, i.e. $N > n + 1$.

By normal form assumption $x_{n+2} > 1$ and $x_{n+1} = [a_{n+1}, x_{n+2}] < a_{n+1} + 1$.

According to theorem 1 (11) only the sharp inequalities remain and by further use:

1. For $d = (-1)^n \Delta_n - (-1)^{n+1} \Delta_{n+1} = (q_{n+1}\, x_{n+2} + q_n) - (q_n\, x_{n+1} + q_{n-1})$ we get

$d > (q_{n+1} + q_n) - (q_n\, a_{n+1} + q_n + q_{n-1}) = 0$

2. Set $h = (-1)^n ((x - A_n) - (A_{n+1} - x))$. Then $h = q_{n+1}\,(q_{n+1}\, x_{n+2} + q_n) - q_n\,(q_n\, x_{n+1} + q_{n-1})$

and as in (1) $h > q_{n+1}\,(q_{n+1} + q_n) - q_n\,(q_{n+1} + q_n) = q_{n+1}^2 - q_n^2 \geq 0$.

The next theorem is only needed to give a special form of a later condition for best-convergents.

Theorem. Define polynomials $P_n(a_0, \ldots, a_n)$ and $Q_n(a_0, \ldots, a_n)$ by

$\begin{pmatrix} P_{-2} \\ Q_{-2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} P_n \\ Q_n \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $n \geq -1$

Then

1. $\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for $n \geq -1$
2. $P_n(a_0, \ldots, a_n) = P_n(a_n, \ldots, a_0)$ for $n \geq -1$

   $Q_n(a_n, \ldots, a_0) = P_{n-1}(a_0, \ldots, a_{n-1})$ for $n \geq 0$

3. $Q_n(a_0, \dots, a_n) = P_{n-1}(a_1, \dots, a_n)$ for $n \geq 0$
4. $[a_0, \dots, a_n] = P_n(a_0, \dots, a_n)/P_{n-1}(a_1, \dots, a_n)$ for $n \geq 0$
5. $Q_{n+1}(a_0, \dots, a_{n+1})/Q_n(a_0, \dots, a_n) = [a_{n+1}, \dots, a_1]$ for $n \geq 0$

Proof: Direct verifications for the lowest values of n are necessary and assumed.
1. As $\begin{pmatrix} P_{n-1} \\ Q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $n \geq 0$
2. By transposing (1), which gives $\begin{pmatrix} P_n & Q_n \\ P_{n-1} & Q_{n-1} \end{pmatrix} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}$
3. From (2)
4. From (3)
5. $Q_{n+1}(a_0, \dots, a_{n+1})/Q_n(a_0, \dots, a_n) = P_n(a_1, \dots, a_{n+1})/P_{n-1}(a_1, \dots, a_n) =$
   $P_n(a_{n+1}, \dots, a_1)/P_{n-1}(a_n, \dots, a_1) = [a_{n+1}, \dots, a_1]$ the last by (4)

Construction of Continued Fractions

Theorem. Construct for $y \in \mathbb{R}$ a simple integer continued fraction expansion $x = [a_0, a_1, ..., a_n, \dots]$ by induction:

$y_0 = y,\ a_0 = \lfloor a \rfloor,$

if $y_0 > a_0 : y_1 = 1/(y_0 - a_0),\ a_1 = \lfloor y_1 \rfloor, \dots$

if $y_n > a_n : y_{n+1} = 1/(y_n - a_n),\ a_{n+1} = \lfloor y_{n+1} \rfloor \dots$

As x now is defined the previous notation and results may be applied. We have
1. The construction never gives long form, and all $a_{n+1} \geq 1$.¤
2. $y = [a_0, ..., a_{n-1}, y_n]$ and $y_n = \langle x_n \rangle$ or just $y_n = x_n$ by theorem 1 (10)
3. If $y_n = a_n$, then the expansion ends and $\langle x \rangle = A_n$.
   Otherwise the expansion is infinite and $y = \langle x \rangle$

Proof: 1. As $0 < y_n - \lfloor y_n \rfloor < 1 \Rightarrow y_{n+1} > 1 \wedge a_{n+1} \geq 1$ and $y_{n+1} \in \mathbb{Z} \Rightarrow a_{n+1} \geq 2$.
2. From $y = [y_0]$ and $y_{n+1} = 1/(y_n - a_n) \Leftrightarrow y_n = [a_n, y_{n+1}]$, we have
$y = [a_0, ..., a_{n-1}, y_n] \Rightarrow y = [a_0, ..., a_{n-1}, a_n, y_{n+1}]$, and by theorem 1 (10) $y_n = \langle x_n \rangle$
3. For an infinite expansion Theorem 1 (3, 5b) gives $\lim_{n\to\infty} A_{2n} \leq x \leq \lim_{n\to\infty} A_{2n+1}$ and $\lim_{n\to\infty} A_n = x$

Convergents

Assume in this paragraph that $x = [a_0, a_1, ..., a_n, \dots]$, $n \geq 0$ is a simple integer continued fraction in normal form.

Theorem (Lagrange). Let $p, q \in \mathbb{Z} \wedge 1 \leq q \leq q_n \wedge n \geq 1 \wedge p/q \neq A_n$. Then
1. $|p_{n-1} - x\, q_{n-1}| \leq |p - x\, q|$, where equality only holds for $p/q = A_{n-1}$
2. $|A_n - x| < |p/q - x|$

Proof: The equations $q = y\, q_n + z\, q_{n-1}$ and $p = y\, p_n + z\, p_{n-1}$ has determinant $(-1)^n$ and solution $y = (-1)^n (q\, p_{n-1} - p\, q_{n-1})$ and $z = (-1)^n (p\, q_n - q\, p_n)$, where $y, z \in \mathbb{Z}$. Now $y z \leq 0$ since otherwise $|q| \geq |q_n| + |q_{n-1}|$. Also $z \neq 0$, as $z = 0 \Rightarrow p/q = A_n$. If $y = 0$, then $p/q = A_{n-1}$, the limit case. For $y, z \neq 0$ by $x\, q - p = y\ \Delta_n + z\, \Delta_{n-1}$ and sign considerations, $|x\, q - p| \geq |\Delta_n| + |\Delta_{n-1}| > |\Delta_{n-1}|$.
Finally (2) follows from $|p/q - x|\, q \geq |\Delta_{n-1}| > |\Delta_n| = |A_n - x|\, q_n \geq |A_n - x|\, q$.

Corollary. If $x = p/q \in \mathbb{Q}$ the continued fraction is finite.

Proof: If it is infinite choosing A_n with $q_n > q$ gives a contradiction to the theorem 4 (2).

Definition. Let $p, q, r, s \in \mathbb{Z} \wedge 1 \le q, s$. Then

p/q is a best approximation of second kind to x, if $s \le q \Rightarrow \left(|p - x\,q| < |r - x\,s| \vee p/q = r/s\right)$

Obviously a best p/q is irreducible.

In the case $q = 1$ we have (i) $x - a_0 < ½ \Rightarrow a_0$ best,

(ii) $x - a_0 = ½ \Rightarrow$ no best, (iii) $x - a_0 > ½ \Rightarrow a_0 + 1$ best

Theorem. Let $p, q \in \mathbb{Z}$, $1 \le q$. Then

p/q is a best approximation of second kind to $x \Leftrightarrow p/q$ is a convergent to x .

Proof: $\Rightarrow$: As $q_{n-1} \le q < q_n$ for some $n \ge 1$, using the approximation condition with $r = p_{n-1}$, $s = q_{n-1}$ implies $p/q = r/s$ by theorem 4 (1).

$\Leftarrow$: From Lagrange's theorem 4 (1) with $p = r \wedge q = s$ follows

$|p_n - x\,q_n| = |\Delta_n| < |\Delta_{n-1}| \le |r - x\,s|$ for $1 < s \le q_n \wedge r/s \ne A_n$.

This shows A_n with $n \ge 1$ is a best approximation of second kind.

Example. $7/9$ has convergents $0 = [0]$, $1 = [0, 1]$, $3/4 = [0, 1, 3]$, $7/9 = [0, 1, 3, 2]$

and the last three as best approximations of second kind.

If long form is used, then $7/9 = [0, 1, 3, 1, 1]$ has convergents

$3/4 = [0, 1, 3]$, $4/5 = [0, 1, 3, 1]$; but $|3 - (7/9)\cdot 4| = |4 - (7/9)\cdot 5| = 1/9$

which shows that $4/5$ is not a best approximation of second kind.

Semiconvergents and Best-convergents

In this paragraph $x = [a_0, a_1, ..., a_n, \ldots]$, $n \ge 0$ need not to be in normal form.

Lemma. For $y, z \ge 0$, $(n, y) \ne (0, 0)$, $(n, z) \ne (0, 0)$ define

$p_{n,z} = p_n\,z + p_{n-1}$, $q_{n,z} = q_n\,z + q_{n-1}$ and $A_{n,z} = p_{n,z}/q_{n,z}$.

Then $A_{n,z} - A_{n,y} = (-1)^n\,(y - z)\,/(q_{n,y}\,q_{n,z})$ and $A_{n,y} - A_n = (-1)^n\,/(q_{n,y}\,q_n)$

Furthermore $y > 0 \Rightarrow A_{n,y} = [a_0, ..., a_n, y]$ and $A_{n+1,0} = A_n = A_{n-1,a_n}$, if defined.

Proof: $A_{n,z} - A_{n,y} = \frac{p_{n,z}}{q_{n,z}} - \frac{p_{n,y}}{q_{n,y}} = \frac{(y-z)\,(p_{n-1}\,q_n - p_n\,q_{n-1})}{q_{n,y}\,q_{n,z}} = \frac{(-1)^n\,(y-z)}{q_{n,y}\,q_{n,z}}$

The next follows likewise, and the remaining by theorem 1.

We now collect some inequalities in the next lemma

Lemma. 1. For $y > z \ge 0$, $(n, z) \ne (0, 0)$ holds $(-1)^n A_{n,y} < (-1)^n A_{n,z}$

2. $(-1)^n A_n < (-1)^n x \le (-1)^n A_{n+1} = (-1)^n A_{n,a_{n+1}} < \ldots < (-1)^n A_{n,1}$

Proof: Follows easily from theorem 1 and lemma 1.

Theorem. Suppose $m \in \{1, \ldots, a_{n+1}\}$.

Define a best-convergent $A_{n,m}$ as those fractions, for which $|A_{n,m} - x| < |A_n - x|$ and $q_{n,m} \ge 2$. Then we have

1. $|A_{n,m} - x| < |A_n - x| \Leftrightarrow x_{n+1} < 2\,m + q_{n-1}/q_n$.

2. Long and for normal form gives the same total set of best-convergents.

3. The best-convergents values $|A_{n,m} - x|$ are strictly lexicographic opposite ordered:
$|A_{k,i} - x| > |A_{n,m} - x| \Leftrightarrow (k < n) \bigvee (k = n \bigwedge i < m)$

Proof: 1. $A_{n,m} - x = A_{n,m} - A_{n,x_{n+1}} = (-1)^n (x_{n+1} - m) / (q_{n,m}\, q_{n,x_{n+1}})$
Hence $|A_{n,m} - x| < |A_n - x| \Leftrightarrow$
$\frac{x_{n+1}-m}{q_{n,m}(q_n x_{n+1}+q_{n-1})} < \frac{1}{q_n (q_n x_{n+1}+q_{n-1})} \Leftrightarrow x_{n+1} < 2\,m + \frac{q_{n-1}}{q_n}$
2. In $x = [a_0,\ a_1,\ \ldots,\ a_N - 1,\ 1] = [b_0,\ b_1,\ \ldots,\ b_{N+1}]$ we may suppose $N \geq 1$.
Best-convergents of the long form are denoted $B_{n,m}$. If $n \leq N - 1$ we get the same as before, except A_{N-1,a_N} is missing, but this fraction is just $B_{N,1}$.
3. By last lemma $i < m \Rightarrow |A_{n,m} - x| < |A_{n,i} - x|$ and
$k < n \Rightarrow |A_{n,m} - x| < |A_n - x| \leq |A_{k+1} - x| = \left|A_{k,a_{k+1}} - x\right| \leq |A_{k,i} - x|$

Corollary. Let $m \in \mathbb{Z}_+$ and $m \in \{1,\ \ldots,\ a_{n+1}\}$. Then
1. (i) $m > a_{n+1}/2 \Rightarrow A_{n,m}$ is a best-convergent
(ii) For $m = a_{n+1}/2$: $A_{n,m}$ is a best-convergent $\Leftrightarrow x_{n+1} < q_{n+1}/q_n \Leftrightarrow [a_{n+1},\ a_{n+2},\ \ldots] < [a_{n+1},\ \ldots,\ a_1]$.
(iii) $m < a_{n+1}/2 \Rightarrow A_{n,m}$ is not a best-convergent
2. Set $\beta_n = \begin{cases} a_{n+1}/2, & \text{if } a_n \text{ is even and } x_{n+1} < q_{n+1}/q_n \\ \lfloor (a_{n+1}+1)/2 \rfloor & \text{otherwise} \end{cases}$. Then
$A_{n,m}$ is a best-convergent $\Leftrightarrow m \in \{\beta_n,\ \ldots,\ a_{n+1}\}$

Proof: 1. Follows easily from expressing theorem 6 (1) in cases:
(i) If $a_{n+1} \leq 2\,m - 1$ then $x_{n+1} < a_{n+1} + 1 \leq 2\,m + q_{n-1}/q_n$, thus $|A_{n,m} - x| < |A_n - x|$
(ii) If $a_{n+1} = 2\,m$ then $|A_{n,m} - x| < |A_n - x| \Leftrightarrow$
$x_{n+1} < a_{n+1} + q_{n-1}/q_n \Leftrightarrow x_{n+1} < (q_n\, a_{n+1} + q_{n-1})/q_n \Leftrightarrow x_{n+1} < q_{n+1}/q_n$
The last follows from theorem 1 (10) and theorem 2 (5)
(iii) As $a_{n+1} \geq 2\,m + 1 \Rightarrow x_{n+1} \geq 2\,m + 1$, we get $|A_{n,m} - x| \geq |A_n - x|$
2. Using (1) and setting β_n to the lowest m, for which $A_{n,m}$ is a best-convergent

Lemma. Let $m \geq 1$, $(n,\ m) \neq (0,\ 1)$ and $p,\ q \in \mathbb{Z}$, $q \geq 1$.
1. If $(-1)^n A_{n,m} < (-1)^n p/q < (-1)^n A_{n,m-1}$, then $q > q_{n,m}$
2. If $(-1)^n A_n < (-1)^n p/q < (-1)^n A_{n,m}$, then $q > q_{n,m}$

Proof: We shall use lemma 1 and lemma 2.
1. From $0 < |A_{n,m-1} - p/q| < |A_{n,m} - A_{n,m-1}| = 1/(q_{n,m-1}\, q_{n,m})$ follows
$0 < |q\, p_{n,m-1} - p\, q_{n,m-1}| < q/q_{n,m}$. Hence $1 < q/q_{n,m}$.
2. As $A_n = \lim_{m\to\infty} A_{n,m}$ monotonically, we get for some $k \geq m + 1$ that $(-1)^n A_{n,k} \leq (-1)^n p/q < (-1)^n A_{n,k-1}$.
Thus $q \geq q_{n,k} > q_{n,m}$.

Corollary. Let $A_{n,m}$ be a best-convergent, $p,\ q \in \mathbb{Z}$, $1 \leq q \leq q_{n,m}$.
Then $|A_{n,m} - x| \leq |p/q - x|$, where equality only holds for $p/q = A_{n,m}$.

Proof: The proof is done by the cases
(i) $(-1)^n p/q \leq (-1)^n A_n$, giving $|A_{n,m} - x| < (-1)^n (x - A_n) \leq (-1)^n (x - p/q)$
(ii) $(-1)^n A_n < (-1)^n p/q < (-1)^n A_{n,m}$, giving $q > q_{n,m}$

(iii) $(-1)^n A_{n,m} < (-1)^n p/q$, implying $0 \le (-1)^n (A_{n,m} - x) < (-1)^n (p/q - x)$
(iv) $(-1)^n A_{n,m} = (-1)^n p/q$

Definition. Let $p, q, r, s \in \mathbb{Z} \wedge 1 \le q, s$ and $p/q \ne r/s$. Then
p/q is a best approximation of first kind to x, if $s \le q \Rightarrow |p/q - x| < |r/s - x|$
Obviously a best p/q is irreducible.
In the case $q = 1$ the result is the same as for second kind (i) $x - a_0 < ½ \Rightarrow a_0$ best,
(ii) $x - a_0 = ½ \Rightarrow$ no best, (iii) $x - a_0 > ½ \Rightarrow a_0 + 1$ best

Theorem. Let $p, q \in \mathbb{Z}$, $2 \le q$ and p/q irreducible. Then
p/q is a best approximation of first kind to $x \Leftrightarrow p/q$ is a best-convergent to x

Proof: $\Leftarrow$: Let $p/q = A_{n,m}$ be a best-convergent and $s \le q_{n,m}$. Then by the corollary
$r/s \ne A_{n,m} \Rightarrow |A_{n,m} - x| < |r/s - x|$. This shows $A_{n,m}$ is a best approximation of first kind.
$\Rightarrow$: 1. Assume p/q is not a semiconvergent or a convergent. As $A_0 = a_0 \le x < A_{0,1} = a_0 + 1$, the cases $p/q < a_0$ and $p/q > a_{0,1} = a_0 + 1$ are impossible because one of A_0, $A_{0,1}$ are closer to x than p/q and has denominator 1. From lemma 2
$A_0 = A_{1,0} < A_{1,1} \ldots < A_{1,a_2} = A_2 = A_{3,0} < A_{3,1} < \ldots x \ldots < A_{2,1} < A_{2,0} = A_1 = A_{0,a_1} < \ldots < A_{0,1}$
follows $(-1)^n x \le (-1)^n A_{n,m} < (-1)^n p/q < (-1)^n A_{n,m-1}$ for some n, m, and $m = 1$ is allowed, if $n > 0$. This is the main trick.
It implies $|A_{n,m} - x| \le |p/q - x|$, and also by lemma 3 (1) $q > q_{n,m}$, which gives a contradiction, if $r/s = A_{n,m}$ is used in the definition of best approximation
2. By (1) p/q is a semiconvergent or a convergent, and $p/q = A_{n,m}$ for some n, m, as $q \ge 2$.
If $p/q = A_{n,m}$, we have $|A_{n,m} - x| < |A_n - x|$, as p/q is a best approximation. Hence p/q is a best-convergent.

Example. $x = \left(1 + \sqrt{2}\right)/2 = [1, 4, 1, 4, 1, 4, \ldots]$ has best-convergents
$1 = [1]$, $4/3 = [1, 3]$, $5/4 = [1, 4]$, $6/5 = [1, 4, 1]$, $23/19 = [1, 4, 1, 3] \ldots$.
Furthermore $17/14 = [1, 4, 1, 2]$ is not a best approximation of first kind, as
$|17/14 - x| > 0.00717 > 0.00711 > |6/5 - x|$
This is confirmed using theorem (1.ii), as $[4, 1, 4, 1, 4, \ldots] > [4, 1, 4]$.

References

[1] Oskar Perron, Die Lehre von den Kettenbrüchen (Teubner 1913)
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