Generalized Spherical Triangles v3

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Abstract

The purpose is to introduce spherical triangles with one ore more angles or sides greater than pi. A signed area theory extends with a little modification the connection between area and spherical excess , and facilitates a classification of the triangles.

The triangle cases are investigated. Formulae for the radii of circumcircles , in- and excircles are found.

Also the assignment problem for incircles is solved. A duality theorem is proved.

Contents

1. The basic theory

By cyclic permutations of *A, B, C* figures 2, 3, 6, 7 has three versions a, b, c. Here only the a-version is shown.

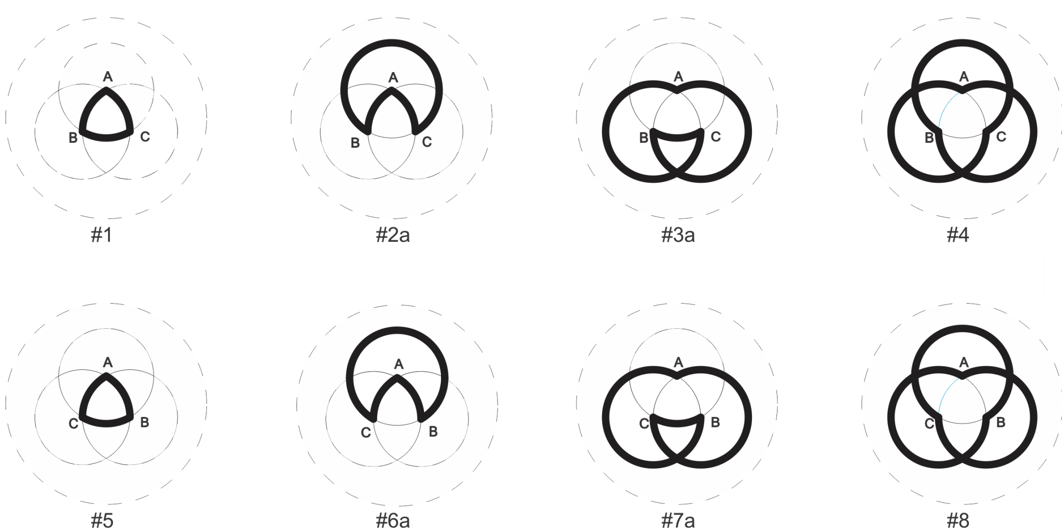

Fig. 1. The **stereographic** images (idealized) illustrates GST types numbered in the table.
Number 1 to 4 has positive- and the rest negative-orientation.
The closed triangle curve has one double-point for types #3, #7, three for #4, #8, and otherwise zero.

In 3D euclidean space three unit vectors **a**, **b**, **c** representing points *A,B,C* not in a plane determines a classical or *elementary* spherical triangle, indicated with Δ as Δ*ABC*. The points *A*, *B*, *C* and their antipodes properly selected in

tripples covers the sphere with 8 elementary triangles.
The triangle orientation, sgn(ABC) is the sign of the determinant of the matrix with the coordinates of **a**, **b**, **c** as columns.

A generalized spherical triangle is given by such three points A,B,C or vertices and three oriented great circle segments BC, CA, AB , named a, b, c, each directed from the first point to the second. By this the three great circles are also oriented.
As each segment can be short or long, i.e. in $(0, \pi)$ or $(\pi, 2\pi)$, there are eight GST's with given vertices A, B, C. Long segments are indicated by dashes as in GST($A - BC -$) meaning long segments AB, CA and a short segment BC.

The angle A (or just A) of a GST is determined as the angle in $(0, 2\pi)$ from the direction of AB at A to the direction of AC at A.
The three great circles that contain the sides of the GST divides the sphere into eight elementary GST's and a orientation of a part of these connected along the sides are possible only in one way, if it shall be consistent with the original GST.

Definition 1.1.
GST involutions:
F_a shifts the side a between long and short by which the adjacent angles are changed π (mod 2π), e.g. GST(ABC) $\to$ GST($AB-C$).
F_b and F_c are likewise defined.
R_o is the reflection in origo mapping a point P into its antipodal point P'. It also maps oriented segments pointwise.
By R_o a GST **X** is mapped onto a GST $-$**X** with vertices $A, B, C \to A', B', C'$ such that

- sides: lengths are unchanged
- angles: $(A, B, C) \to (2\pi - A, 2\pi - B, 2\pi - C)$

We have e.g. that $-$GST($A - BC -$) = GST($A' - B'C' -$).

Algebraic considerations.
We have the obvious

Theorem 6.1. The GST transformations F_a , F_b , F_c , R_o are all idempotent and generates a *commutative* group **G** of order 16 of idempotent *permutations* of the 16 triangles in Table 1, i.e. **G** act as a permutation group on the GST's. Moreover **G** work transitively.

We know that R_o shifts the orientation and the subgroup generated by F_a , F_b , F_c keeps the orientation.

The group **G** is isomorphic to $(\mathbb{Z}_2, \oplus)^4$, where the elements can be written as 4 bit words, and addition $\oplus$ is (mod 2) for each bit, also called XOR .
The isomorphy is generated by $(F_a , F_b , F_c , R_o) \to ((0001)_2, (0010)_2, (0100)_2, (1000)_2).\ )_2$
By $g \to g(T_0)$, $g \in G$ is bijective the triangles in Table 1 is uniquely represented by elements in G or $(\mathbb{Z}_2, \oplus)^4$.
The elements may be written in the 10- number system.

Examples. $F_a F_a \to (0001 \oplus (0001)_2 = (0000)_2 = 0$ representing the identity transformation or triangle T_0
$F_b\ R_o F_a \to (0010)_2 \oplus (1000)_2 \oplus (0001)_2 = (1011)_2 = 11$ representing triangle $F_b\ R_o F_a(T_0)$

Theorem 1.2. For given points ABC with there exist 16 GST's with specifications shown in table 2.
These triangles are classified in eight types by the index (si, ai) = (#sides $> \pi$, #angles $> \pi$).

Case #1 has the same sides and angles as the classical triangles ΔABC and $\Delta A'B'C'$.

Proof:

1. For each side in a GST there is a transformation, e.g. F_a, shifting the side between long and short by which the adjacent angles are changed π (mod 2π).

Theorem 1.2. imply obviously:

Corollary 1.3. A GST triangle is positively oriented $\Leftrightarrow$ the angle index is even

Theorem 1.5. For a GST the cosine- and the sine-relations hold.
Proof: Inserting the variables from the table verifies the laws.

id#	index	notation	a	b	c	A	B	C	$A+B+C$	$Area$	$A+B+C$	$A+B+C-Area$
1	(0, 0)	GST (ABC)	a_0	b_0	c_0	A_0	B_0	C_0	S_0	$T_0 = S_0 - \pi$	$\pi + T_0$	π
2 a	(1, 2)	GST $(AB-C)$	$2\pi - a_0$	b_0	c_0	A_0	$\pi + B_0$	$\pi + C_0$	$2\pi + S_0$	$2\pi + T_0$	$3\pi + T_0$	π
2 b	(1, 2)	GST $(ABC-)$	a_0	$2\pi - b_0$	c_0	$\pi + A_0$	B_0	$\pi + C_0$	$2\pi + S_0$	$2\pi + T_0$	$3\pi + T_0$	π
2 c	(1, 2)	GST $(A-BC)$	a_0	b_0	$2\pi - c_0$	$\pi + A_0$	$\pi + B_0$	C_0	$2\pi + S_0$	$2\pi + T_0$	$3\pi + T_0$	π
3 a	(2, 2)	GST $(A-BC-)$	a_0	$2\pi - b_0$	$2\pi - c_0$	A_0	$\pi + B_0$	$\pi + C_0$	$2\pi + S_0$	T_0	$3\pi + T_0$	3π
3 b	(2, 2)	GST $(A-B-C)$	$2\pi - a_0$	b_0	$2\pi - c_0$	$\pi + A_0$	B_0	$\pi + C_0$	$2\pi + S_0$	T_0	$3\pi + T_0$	3π
3 c	(2, 2)	GST $(AB-C-)$	$2\pi - a_0$	$2\pi - b_0$	c_0	$\pi + A_0$	$\pi + B_0$	C_0	$2\pi + S_0$	T_0	$3\pi + T_0$	3π
4	(3, 0)	GST $(A-B-C-)$	$2\pi - a_0$	$2\pi - b_0$	$2\pi - c_0$	A_0	B_0	C_0	S_0	$T_0 - 2\pi$	$\pi + T_0$	3π
5	(0, 3)	$-$GST (ABC)	a_0	b_0	c_0	$2\pi - A_0$	$2\pi - B_0$	$2\pi - C_0$	$6\pi - S_0$	$4\pi - T_0$	$5\pi - T_0$	π
6 a	(1, 1)	$-$GST $(AB-C)$	$2\pi - a_0$	b_0	c_0	$2\pi - A_0$	$\pi - B_0$	$\pi - C_0$	$4\pi - S_0$	$2\pi - T_0$	$3\pi - T_0$	π
6 b	(1, 1)	$-$GST $(ABC-)$	a_0	$2\pi - b_0$	c_0	$\pi - A_0$	$2\pi - B_0$	$\pi - C_0$	$4\pi - S_0$	$2\pi - T_0$	$3\pi - T_0$	π
6 c	(1, 1)	$-$GST $(A-BC)$	a_0	b_0	$2\pi - c_0$	$\pi - A_0$	$\pi - B_0$	$2\pi - C_0$	$4\pi - S_0$	$2\pi - T_0$	$3\pi - T_0$	π
7 a	(2, 1)	$-$GST $(A-BC-)$	a_0	$2\pi - b_0$	$2\pi - c_0$	$2\pi - A_0$	$\pi - B_0$	$\pi - C_0$	$4\pi - S_0$	$-T_0$	$3\pi - T_0$	3π
7 b	(2, 1)	$-$GST $(A-B-C)$	$2\pi - a_0$	b_0	$2\pi - c_0$	$\pi - A_0$	$2\pi - B_0$	$\pi - C_0$	$4\pi - S_0$	$-T_0$	$3\pi - T_0$	3π
7 c	(2, 1)	$-$GST $(AB-C-)$	$2\pi - a_0$	$2\pi - b_0$	c_0	$\pi - A_0$	$\pi - B_0$	$2\pi - C_0$	$4\pi - S_0$	$-T_0$	$3\pi - T_0$	3π
8	(3, 3)	$-$GST $(A-B-C-)$	$2\pi - a_0$	$2\pi - b_0$	$2\pi - c_0$	$2\pi - A_0$	$2\pi - B_0$	$2\pi - C_0$	$6\pi - S_0$	$2\pi - T_0$	$5\pi - T_0$	3π

Table 2.

2. GST problems and solutions

Solving a GST for three objects of A, B, C, a, b, c known is done by first calculating the known basis variables among A_0, B_0, C_0, a_0, b_0, c_0.
The calculations are based on table 3, where 0 means $< \pi$ and 1 means $> \pi$. The problem type uses letters A and S for known angle and side, eg. SAS means the angle is between the sides, and SSA mean that the angle is opposite to on of the sides.

Theorem 3.1. Each of the six types of a GST problem matches exact two id# in table 3, when looking at the known data. The corresponding orientations is the same for AAA, SAS and opposite for SSS, SSA, SAA, ASA problems.
For each id# data can be transformed to #1 by table 2, to get a reduced problem (RP), and then solved. Any solution found here can be transformed back and give a solution for the GST.

Proof: By looking at the relevant part of table 3 it is obvious.

id#	a	b	c	A	B	C
1	0	0	0	0	0	0
2 a	1	0	0	0	1	1
2 b	0	1	0	1	0	1
2 c	0	0	1	1	1	0
3 a	0	1	1	0	1	1
3 b	1	0	1	1	0	1
3 c	1	1	0	1	1	0
4	1	1	1	0	0	0

Positive Orientation

id#	a	b	c	A	B	C
5	0	0	0	1	1	1
6 a	1	0	0	1	0	0
6 b	0	1	0	0	1	0
6 c	0	0	1	0	0	1
7 a	0	1	1	1	0	0
7 b	1	0	1	0	1	0
7 c	1	1	0	0	0	1
8	1	1	1	1	1	1

Negative Orientation

Table 3 (0 means $< \pi$ and 1 means $> \pi$).

3. Examples

Example 3.1. (SSA): Let $a = 1.4\,\pi,\ b = 2,\ A = 1.3\,\pi$. In table 3 only #3b and #6a are possible.

For #3b-case $a_0 = 2\,\pi - a = 0.6\,\pi,\ b_0 = b,\ A_0 = 1.3\,\pi - \pi = 0.3\,\pi$ and we find

$c_0 = 0.1878465826,\ B_0 = 2.257256214,\ C_0 = 0.1595295613,\ T_0 = 0.2176709175.$

Transforming back to find the GST values:

$c = 2\,\pi - c_0 = 6.095338725,\ B = B_0,\ C = \pi + C_0 = 3.301122215,\ T = T_0.$

For #6a-case $a_0 = 2\,\pi - a = 0.6\,\pi,\ b_0 = b,\ A_0 = 2\,\pi - 1.3\,\pi = 0.7\,\pi$ and we find

$c_0 = 2.006311836,\ B_0 = 2.257256214,\ C_0 = 2.260798179,\ T_0 = 3.575576597$

$c = c_0,\ T = 2.707608711,\ B = \pi - B_0 = 0.8843364399,\ C = \pi - C_0 = 0.8807944746$

Example 3.2. (SAA): Let $a = 0.9\,\pi,\ A = 1.8\,\pi,\ B = 1.6\,\pi$. In table 3 only #2c and #5 are possible.

For #2c-case $a_0 = a,\ A_0 = 1.8\,\pi - \pi = 0.8\,\pi,\ B_0 = 1.6\,\pi - \pi = 0.6\,\pi$ and we find 2 solutions

1) $b_0 = 2.617993878,\ c_0 = 0.5370419222,\ C_0 = 1.338387952,\ T_0 = 2.595025014$ and by table 2

$b = b_0,\ c = 2\,\pi - c_0 = 5.746143385,\ C = C_0,\ T = 2\,\pi + T_0 = 8.878210321,$

2) $b_0 = 0.5235987756,\ c_0 = 2.804691386,\ C_0 = 2.461621775,\ T_0 = 3.718258836$ and by table 2

$b = b_0,\ c = 2\,\pi - c_0 = 3.478493921,\ C = C_0,\ T = 2\,\pi + T_0 = 10.00144414.$

The #5-case $a_0 = a,\ A_0 = 2\,\pi - 1.8\,\pi = 0.2\,\pi,\ B_0 = 2\,\pi - 1.6\,\pi = 0.4\,\pi$ has 0 solutions.

Example 3.3. (SAS): Let $a = 1.4\,\pi,\ b = 1.7\,\pi,\ C = 1.1\,\pi$. In table 3 only #7c and #8 are possible.

For both cases $a_0 = 2\,\pi - 1.4\,\pi = 0.6\,\pi,\ b_0 = 2\,\pi - 1.7\,\pi = 0.3\,\pi$ and $C_0 = 2\,\pi - 1.1\,\pi = 0.9\,\pi,$

which gives $c_0 = 2.72235217,\ A_0 = 0.8066544247,\ B_0 = 0.6613081699,\ T_0 = 1.153803329.$

Transforming back to find the GST values:

#7c: $c = c_0,\ A = \pi - A_0 = 2.334938229,\ B = \pi - B_0 = 2.480284484,\ T = -1.153803329$

#8: $c = 2\,\pi - c_0 = 3.560833137,\ A = 2\,\pi - A_0 = 5.476530882,\ B = 2\,\pi - B_0 = 5.621877137,$

$T = 2\,\pi - T_0 = 5.129381978$

Example 3.4. (AAA): Let $A = 0.1\,\pi,\ B = 0.1\,\pi,\ C = 0.1\,\pi$. In table 3 only #1 and #4 are possible.

For both cases $A_0 = A,\ B_0 = B,\ C_0 = C$ which has no solutions e.g. as $E_0 = -0.7\,\pi < 0$.

4. Theory for the GST cases

The number of solutions of a GST problem is to be investigated.

This number is always counted aside from orientation preserving congruences.

Theorem 4.2. The number of solutions of a GST problem with further details is in table 4 under the assumption that one of the reduced problems has a solution.

type	solutions	conditions	orientations	solutions swapped by
SSS	2		opposite	R_o
AAA	2		same	$F_a \circ F_b \circ F_c$
SAS	2		same	F_a , if SAS = cAb
ASA	2		opposite	for ASA = AbC : $B \to B'$ and o rientations of greatcircles by a, b, c unchanged .
SSA = abA	∞	a, b, A $\pi/2$ or $3\pi/2$	opposite	
	1	$A \neq \pi/2,\ 3\pi/2$ $\lvert\sin b\rvert = \lvert\sin a\rvert$ or $\lvert\sin a\rvert = \lvert\sin A\rvert\ \lvert\sin b\rvert$		
	2	$\lvert\sin a\rvert > \lvert\sin b\rvert$	opposite	
	2	otherwise	same	
SAA = aAB	∞	A, B, a $\pi/2$ or $3\pi/2$	opposite	
	1	$a \neq \pi/2,\ 3\pi/2$ $\lvert\sin B\rvert = \lvert\sin A\rvert$ or $\lvert\sin A\rvert = \lvert\sin a\rvert\ \lvert\sin B\rvert$		
	2	$\lvert\sin A\rvert > \lvert\sin B\rvert$	opposite	
	2	otherwise	same	

Table 4 under the assumption that one of the reduced problems has a solution.

Proof: To get a reduced problem (RP) it is seen from table 2 that sides greater than π always should be subtracted from 2π; but angles are calculated dependent of orientation:

for #1-4: $A_0 = \begin{cases} A & \text{for } A < \pi \\ A - \pi & \text{for } A > \pi \end{cases}$ and similar for B and C, leading to the positive RP (PRP)

for #5-8: $A_0 = \begin{cases} \pi - A & \text{for } A < \pi \\ 2\pi - A & \text{for } A > \pi \end{cases}$ and similar for B and C, leading to the negative RP (NRP).

In all the following cases theorem 3.1 is used.

SSS: The two RP has the same values for a_0, b_0, c_0 giving rise to two GST solutions.

AAA: As the two RP has same orientation they have the same values for A_0, B_0, C_0 giving rise to two GST solutions.

SAS=cAb: A part of a GST an angle A with legs c and b is unique aside from orientation preserving congruences.
This gives us the points A, B, C. Hence two options are left, as a can be long or short.

ASA=AbC:. A side a with angles A and C as part of a GST is unique aside from orientation preserving congruences.
This gives us two possibilities for B as intersection between great circles.
These are antipodal and therefore give opposite orientation for the solutions with unchanged orientations of the great circles$a$, b, c.

SSA=abA: The PRP has start data a_0, b_0, A_0 and the NRP a_0, b_0, $\pi - A_0$.
From a usual spherical problem of type SSA=abA is known that

conditions	solutions	specified
$a,\ b,\ A = \pi/2$	∞	$B = \pi/2,\ C = c$
$\sin a > \sin b$	1	
$\sin b = \sin a$ or $\sin a = \sin A \sin b$ $A,\ a < \pi/2$ or $A,\ a > \pi/2$	1	
$\sin b < \sin a < \sin A \sin b$ $A,\ a < \pi/2$ or $A,\ a > \pi/2$	2	

This result is used with the found start data.

To specify: 1. Let $a = \frac{3\pi}{2}$, $b = A = \frac{\pi}{2}$. Then start data for PRP and NRP coincide.

The positive oriented solutions are found from #2a as

$$a = \tfrac{3\pi}{2},\ b = \tfrac{\pi}{2},\ c = t,\ A = \tfrac{\pi}{2},\ B = \tfrac{3\pi}{2},\ C = \pi + t \text{ with } t \in (0, \pi),$$

and the negative oriented solutions are found from #7b as

$$a = \tfrac{3\pi}{2},\ b = \tfrac{\pi}{2},\ c = 2\pi - t,\ A = \tfrac{\pi}{2},\ B = \tfrac{3\pi}{2},\ C = \pi - t \text{ with } t \in (0, \pi).$$

2. The condition $\sin a_0 > \sin b_0$ is equivalent to $|\sin a| > |\sin b|$,
as $\sin a_0 = |\sin a_0| = |\sin(2\pi - a_0)| = |\sin a|$ and similar for b.
Hence one solution in the PRP and in the NRP.
3. In the remaining two cases for start data the condition $A,\ a < \pi/2$ or $A,\ a > \pi/2$ is fulfilled for precisely one orientation.
From PRP $A_0 = A - \pi$ or A and NRP $A_0 = 2\pi - A$ or $\pi - A$. Hence $\sin A_0 = |\sin A|$, and as just seen $\sin a_0 = |\sin a|$, $\sin b_0 = |\sin b|$.
Therefore e.g. $\sin a_0 = \sin A_0 \sin b_0$ is equivalent to $|\sin a| = |\sin A|\ |\sin b|$

SAA=ABa: The treatment is completely dual to the SSA case.

5. The Circumcircle

Definition 5.1. The circumcircle for a GST has radius given by

$$\tan R = \sqrt{-\cos S/(\cos(S-A)\cos(S-B)\cos(S-C))} \text{ and } R \in (0, \pi/2)$$

and the same center as the classical spherical triangle determined by the vertices. [1]

Theorem 5.2. R is the same for all cases in table 2, and the circumcircle passes through the vertices of the GST.

Proof: Let $G(A, B, C) = -\cos S/(\cos(S-A)\cos(S-B)\cos(S-C))$.
A 2π-change of a single variable in G has no effect, since all the cos-items change sign. Therefore it is sufficient to verify that G is invariant to F_a, F_b, F_c, R_o. This is illustrated in table 5.

#	A	B	C	S	$S-A$	$S-B$	$S-C$	$\cos S$	$\cos S_A$	$\cos S_B$	$\cos S_C$	G
1	A_0	B_0	C_0	S_0	S_A	S_B	S_C	C_S	C_A	C_B	C_C	G
2 a	A_0	$\pi + B_0$	$\pi + C_0$	$\pi + S_0$	$\pi + S_A$	S_B	S_C	$-C_S$	$-C_A$	C_B	C_C	G
5	$2\pi - A_0$	$2\pi - B_0$	$2\pi - C_0$	$3\pi - S_0$	$\pi - S_A$	$\pi - S_B$	$\pi - S_C$	$-C_S$	$-C_A$	$-C_B$	$-C_C$	G

Table 5.

6. Area

The inner part of a GST are illustrated by fig. 1 with color, where the blue marks positive and red marking negative area density.

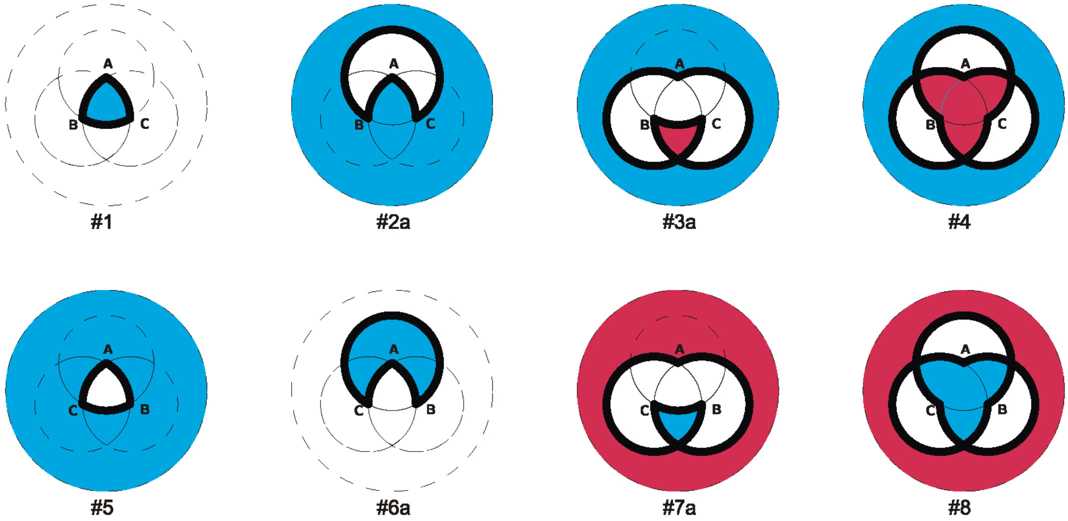

Fig. 1-sec.6. The **stereographic** images (idealized) illustrates GST types numbered in the table.
Number 1 to 4 has positive- and the rest negative-orientation.
The closed triangle curve has one double-point for types #3, #7, three for #4, #8, and otherwise zero.

Theorem. The area of the of a GST is uniquely determined by

1. The open connected parts of a side separates inner and outer parts of a GST.

2a. In each open connected parts of the inner parts of a GST the area of each elementary triangles are taken with some and the same sign. At least one part has has sign as the GST.

2b. If there are 2 open connected parts of the inner parts of a GST the areas of these have opposite signs.

3. The total area of the inner of a GST is $\sigma\, T_0 + h\,\pi$, where T_0 is the usual area, h is an integer, σ the sign of the orientation of the GST.

Condition 2b is for the cases #4 and #8 to ensure a simple extension of the classical area formula:

Theorem 1.4. For a GST the area T is

$$T = \begin{cases} A + B + C - \pi & \text{if the number sides greater than } \pi \text{ is } < 2 \\ A + B + C - 3\,\pi & \text{if the number sides greater than } \pi \text{ is } \geq 2 \end{cases}$$

Lemma. Let f_a be the function on the sphere that are 1 for the closed half-sphere containing A, limited by the great circle a, and 0 elsewhere. Analogously defined are f_b, f_c. Then we have aside from boundaries the signed characteristic functions for the GST's, h_x, where x is the id number of the triangle:

$$h_1 = f_a\, f_b\, f_c\,,\; h_{2\,a} = f_a\, f_b\, f_c - f_a + 1\,,\;\; h_{3\,a} = f_a\, f_b\, f_c - f_b - f_c + 1,$$

$$h_4 = f_a\, f_b\, f_c - f_a - f_b - f_c + 1,\;\; h_5 = 1 - h_1,\; h_{6\,a} = 1 - h_{2\,a},\;\; h_{7\,a} = -h_{3\,a}\,,\; h_8 = -h_4\,.$$

Thus the area e.g. for #3a is $T_0 - 2\,\pi - 2\,\pi + 4\,\pi = T_0$.

It all can be verified from Tabel 2.

7. In- and Excircles.

Background.

For GST of type #1 as a usual spherical triangle ΔABC the incircle is named $C_{0\,i}$.

The excircle $C_{0\,a}$ for ΔABC is the incircle for $\Delta A'BC$ (see fig. 11 #1), and correspondingly for $C_{0\,b}$, $C_{0\,c}$. The circles

$(C_{0\,i}, C_{0\,a}, C_{0\,b}, C_{0\,c})$ has radii a, and by reflection in origo images named $(-C_{0\,i}, -C_{0\,a}, -C_{0\,b}, -C_{0\,c})$.

The radii $(r_{0\,i}, r_{0\,a}, r_{0\,b}, r_{0\,c})$ can be determined by

$$\tan r_{0\,i} = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}\ ,\ r_{0\,i} \in (0, \pi/2),$$

$$\tan r_{0\,a} = \sqrt{\frac{\sin s\sin(s-b)\sin(s-c)}{\sin(s-a)}}\ ,\ r_{0\,a} \in (0, \pi/2)$$ and analogously for $r_{0\,b}, r_{0\,c}$ [1].

All these circles touches the three great circles of the triangle.

Definition 7.1. The in- and ex-circles for a GST are named (C_i, C_c, C_b, C_a), and defined by three *principles*

1. The radii are given by formulas

   for C_i by $\tan r_i = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}\ ,\ r_i \in (0, \pi/2)$,

   for C_a by $\tan r_a = \sqrt{\frac{\sin s\sin(s-b)\sin(s-c)}{\sin(s-a)}}\ ,\ r_a \in (0, \pi/2)$ and likewise for C_b and C_c.

2. C_i is contained in a positive elementary triangle (blue on fig. 1-sec.6).
3. **G** act as a permutation group on the 16 GST's with vertices ABC including their in- and excircles.

Theorem 7.2. The in- and ex-circles are uniquely defined.

An excircle and the incircle touches the same elementary segment.

Proof:

1. A 2π-change of the basic variable sin the four formulas for radii has no effect, since all the sin-items change sign. Also R_o fixes the sides and therefore all the sin-items and the radii.

Thus the transformation expressions follows by symmetry and the transformation law for F_a, which are illustrated in table 6.

We observe the effect of F_a is a permutation of the radii, which then must be true for all elements in **G**. The result is shown in table 7.

	a	b	c	s	$s-a$	$s-b$	$s-c$	$\sin s$	$\sin s_a$	$\sin s_b$	$\sin s_c$	r_i	r_a	r_b	r_c
$\boldsymbol{X}$	a	b	c	s	s_a	s_b	s_c	D_0	D_a	D_b	D_c	r_{xi}	r_{xa}	r_{xb}	r_{xc}
$F_a(\boldsymbol{X})$	$2\pi-a$	b	c	$\pi+s_a$	$s-\pi$	$\pi-s_c$	$\pi-s_b$	$-D_a$	$-D_0$	D_c	D_b	r_{xa}	r_{xi}	r_{xc}	r_{xb}

Table 6.

id#	1	2 a	2 b	2 c	3 a	3 b	3 c	4	5	6 a	6 b	6 c	7 a	7 b	7 c	8
	r_i	r_a	r_b	r_c	r_a	r_b	r_c	r_i	r_i	r_a	r_b	r_c	r_a	r_b	r_c	r_i
	r_a	r_i	r_c	r_b	r_i	r_c	r_b	r_a	r_a	r_i	r_c	r_b	r_i	r_c	r_b	r_a
	r_b	r_c	r_i	r_a	r_c	r_i	r_a	r_b	r_b	r_c	r_i	r_a	r_c	r_i	r_a	r_b
	r_c	r_b	r_a	r_i	r_b	r_a	r_i	r_c	r_c	r_b	r_a	r_i	r_b	r_a	r_i	r_c

Table 7.

2. From table 7, in- and ex-circles principle (2) and fig. 1-sec.6 the incircles are uniquely determined as shown in table 8.

#1	1	2 a	2 b	2 c	3 a	3 b	3 c	4	5	6 a	6 b	6 c	7 a	7 b	7 c	8
	$C_{0\,i}$	$C_{0\,a}$	$C_{0\,b}$	$C_{0\,c}$	$-C_{0\,a}$	$-C_{0\,b}$	$-C_{0\,c}$	$-C_{0\,i}$	$-C_{0\,i}$	$-C_{0\,a}$	$-C_{0\,b}$	$-C_{0\,c}$	$C_{0\,a}$	$C_{0\,b}$	$C_{0\,c}$	$C_{0\,i}$

Table 8.

3. If **G** is represented as $(\mathbb{Z}_2, +)^4$, we observe in table 8 that the set of elements in **G** that fixes $C_{0\,i}$ (called the stabilizer of $C_{0\,i}$ in **G**) is the subgroup **H** = {0000, 1111}, which by the commutativity of **G** then fixes all circles.
The orbit of **G** can now be parametrized from $C_{0\,i}$ by **G/H** shown in table 9, where for each circle an element in the side class has been selected such that the most significant bit corresponds to negative sign in the circle symbol.
The necessary verification of the consistency of table 8 and table 9 is easily done.
As all 8 circles are represented in **G/H**, their images are given by the natural action of **G**.

Example. Triangle #6a or 1001 has incircle $1001 \oplus \mathbf{H} = \{1001,\ 0110\}$ i.e. $-C_{0\,a}$. The excircle C_a for this GST is found from $C_{0\,a}$ or 0001 as $0001 \oplus 1001 \oplus \mathbf{H} = \{1000,\ 0111\}$ or $-C_{0\,i}$.

Remark: If the system for the radii is to be expressed, it correspond to extend **H** to $\boldsymbol{H}_r = \{0000,\ 0111,\ 1000,\ 1111\}$ which fixes $\{\pm C_{0\,i}\}$. This will again give table 7.
Working with the smaller group **H** just gives the sign refinement.

4. As each circle is an incircle in one of the 8 elementary triangles, the last remark in theorem 7.2 is evident from fig. 14.

#	**G** *as 4 bits*	*I*	C_x *as 4 bits*
1	0000	$C_{0\,i}$	0000
2 a	0001	$C_{0\,a}$	0001
2 b	0010	$C_{0\,b}$	0010
2 c	0100	$C_{0\,c}$	0100
3 a	0110	$-C_{0\,a}$	1001
3 b	0101	$-C_{0\,b}$	1010
3 c	0011	$-C_{0\,c}$	1100
4	0111	$-C_{0\,i}$	1000
5	1000	$-C_{0\,i}$	1000
6 a	1001	$-C_{0\,a}$	1001
6 b	1010	$-C_{0\,b}$	1010
6 c	1100	$-C_{0\,c}$	1100
7 a	1110	$C_{0\,a}$	0001
7 b	1101	$C_{0\,b}$	0010
7 c	1011	$C_{0\,c}$	0100
8	1111	$C_{0\,i}$	0000

Table 9

G *as 4 bits*				
0000	C_i	C_a	C_b	C_c
0001	C_a	C_i	$-C_c$	$-C_b$
0010	C_b	$-C_c$	C_i	$-C_a$
0100	C_c	$-C_b$	$-C_a$	C_i
0110	$-C_a$	$-C_i$	C_c	C_b
0101	$-C_b$	C_c	$-C_i$	C_a
0011	$-C_c$	C_b	C_a	$-C_i$
0111	$-C_i$	$-C_a$	$-C_b$	$-C_c$
1000	$-C_i$	$-C_a$	$-C_b$	$-C_c$
1001	$-C_a$	$-C_i$	C_c	C_b
1010	$-C_b$	C_c	$-C_i$	C_a
1100	$-C_c$	C_b	C_a	$-C_i$
1110	C_a	C_i	$-C_c$	$-C_b$
1101	C_b	$-C_c$	C_i	$-C_a$
1011	C_c	$-C_b$	$-C_a$	C_i
1111	C_i	C_a	C_b	C_c

Table 10.

The In- and Excircles Result.

In table 10 the complete set of in- and ex-circles is shown starting from an arbitrary GST.
An illustration of the circle positions is on fig. 11.

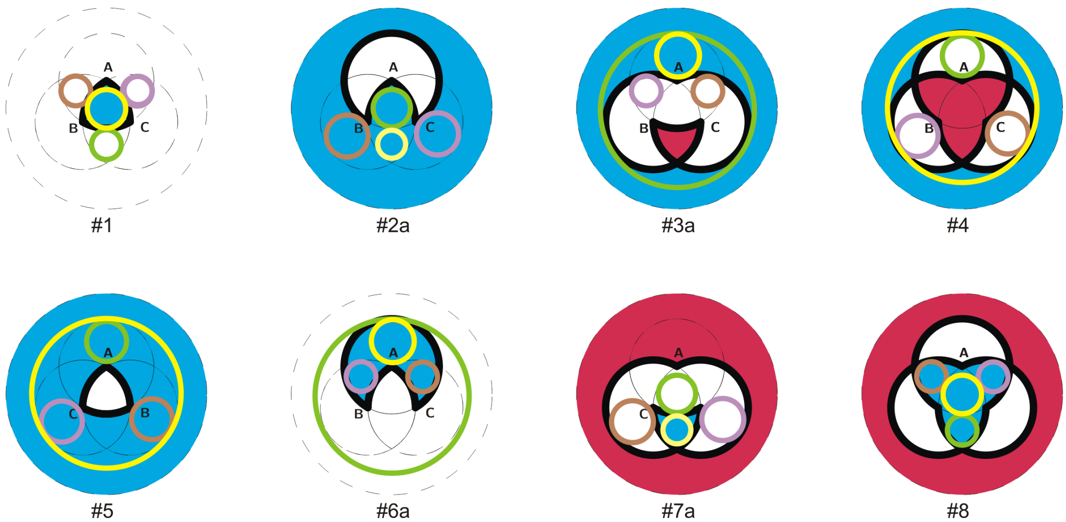

Fig 11. Incircles (yellow) and Excircles, Ca (green), if sgn(ABC)>0 RET ogs 6a

8. Duality.

Definition 8.1. To a GST with elements a, b, c, A, B, C corresponds a polar GST with elements a^*, b^*, c^*, A^*, B^*, C^*, where a^*, b^*, c^* represents the sides, in the following way:
Let vectors unit $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ represent vertices A, B, C and $\boldsymbol{e}_1{}^*$, $\boldsymbol{e}_2{}^*$, $\boldsymbol{e}_3{}^*$ the reciprocal basis given by $\boldsymbol{e}_i{}^* \cdot \boldsymbol{e}_j = 1$ for $i = j$ and otherwise 0. Then $\boldsymbol{e}_1{}^*$, $\boldsymbol{e}_2{}^*$, $\boldsymbol{e}_3{}^*$represent the vertices A^*, B^*, C^*.
Observe that A^*, B^*, C^* also is used as symbols for the angles in the polar (or dual) triangle.
Finally a side is selected long, iff the corresponding side in the original triangle is long.

Theorem 8.2 (Duality).

1. The polar to a GST has the same index and orientation as the original.
2. $a^* > \pi$ iff $a > \pi$ and $A^* > \pi$ iff $A > \pi$
3. The polar to the polar is the original.
4. The duality equations are given in table 13 for a, A.

Proof: 1. The polar to GST$_+$(ABC) is a positive oriented version of the usual polar.
Building the 16 GST's from this triangle satisfies the selection principle in the definition, and also that the polar vertices to A', B', C' are $A^*{}'$, $B^*{}'$, $C^*{}'$.
2. The statement for the sides follows e.g. from table 12 extended to the 16 cases.
3. Follows from (2) and the reciprocal basis to a reciprocal basis is the original.
4. The duality equations are easily deduced from table 12.

GST							Polar GST						
a	b	c	A	B	C	**#**	a^*	b^*	c^*	A^*	B^*	C^*	**#**
a_0	b_0	c_0	A_0	B_0	C_0	1	$\pi - A_0$	$\pi - B_0$	$\pi - C_0$	$\pi - a_0$	$\pi - b_0$	$\pi - c_0$	1
$2\pi - a_0$	b_0	c_0	A_0	$\pi + B_0$	$\pi + C_0$	2 a	$\pi + A_0$	$\pi - B_0$	$\pi - C_0$	$\pi - a_0$	$2\pi - b_0$	$2\pi - c_0$	2 a
a_0	$2\pi - b_0$	$2\pi - c_0$	A_0	$\pi + B_0$	$\pi + C_0$	3 a	$\pi - A_0$	$\pi + B_0$	$\pi + C_0$	$\pi - a_0$	$2\pi - b_0$	$2\pi - c_0$	3 a
$2\pi - a_0$	$2\pi - b_0$	$2\pi - c_0$	A_0	B_0	C_0	4	$\pi + A_0$	$\pi + B_0$	$\pi + C_0$	$\pi - a_0$	$\pi - b_0$	$\pi - c_0$	4
a_0	b_0	c_0	$2\pi - A_0$	$2\pi - B_0$	$2\pi - C_0$	5	$\pi - A_0$	$\pi - B_0$	$\pi - C_0$	$\pi + a_0$	$\pi + b_0$	$\pi + c_0$	5
$2\pi - a_0$	b_0	c_0	$2\pi - A_0$	$\pi - B_0$	$\pi - C_0$	6 a	$\pi + A_0$	$\pi - B_0$	$\pi - C_0$	$\pi + a_0$	b_0	c_0	6 a
a_0	$2\pi - b_0$	$2\pi - c_0$	$2\pi - A_0$	$\pi - B_0$	$\pi - C_0$	7 a	$\pi - A_0$	$\pi + B_0$	$\pi + C_0$	$\pi + a_0$	b_0	c_0	7 a
$2\pi - a_0$	$2\pi - b_0$	$2\pi - c_0$	$2\pi - A_0$	$2\pi - B_0$	$2\pi - C_0$	8	$\pi + A_0$	$\pi + B_0$	$\pi + C_0$	$\pi + a_0$	$\pi + b_0$	$\pi + c_0$	8

Table 12. Duality

9. Miscellaneous

1. The function $G(A,\ B,\ C) = \sqrt{-\cos S/(\cos(S-A)\cos(S-B)\cos(S-C))}$ is tan R for #1, invariant to F_a, F_b, F_c, R_o, also $G(A,\ B,\ C) = G(A \pm 2\pi,\ B,\ C)$ and likewise for B and C, which makes G invariant to **G** transformations.

2b. The formula $F(a,\ b,\ c,\ A,\ B,\ C) = -\tan(a/2)\tan(b/2)\tan(c/2)/\cos(S)$ gives $\tan R$ for case #1 and is invariant to F_a, F_b, F_c, R_o; but gives $-\tan R$ e.g. for #3a. We conclude that F is not a function of the variables (mod 2π). This is confirmed by e.g. $F(a,\ b,\ c,\ A,\ B,\ C) = -F(a + 2\pi,\ b,\ c,\ A,\ B,\ C)$.

3. Let P' be the antipodal point to a point P. By a GST the sphere is divided in eight elementary triangles having vertices in A, B, C and their antipodal points. Half of these are mapped by R_o into the other half. Hence A, B, C, A', B', C' results generally in $8 \cdot 16 = 64$ different GST's.

4. Let the D be the polar transformation of a GST. Then D commutes with **G**.

   Proof: This follows from the construction in the proof of the duality theorem, as a dual to a GST is defined by transforming to #1 by some $\varphi \in$ **G**, taking polar and then again transforming by φ.

5. *The GST Identity Theorem.* Two GST are identical, if they have the same ordered set of vertices and the same pattern for 4 elements in sense of table 3.

   Proof: Each pattern in table 3 for each orientation is unique for the selected 4 elements. Thus the two triangles are transformed by the same $\varphi \in$ **G** from two triangles of type #1, and these are identical as uniquely determined from the vertices.

6. The transformation G_A of a GST is defined to have vertices $A',\ B,\ C$ and unchanged orientations of the great circles a, b, c. Obviously $G_A : (a,\ b,\ c,\ A,\ B,\ C) \rightarrow (a,\ \pi + b,\ \pi + c,\ 2\pi - A,\ B,\ C)$ (mod 2π) (see fig. 13). Also G_A commutes with **G** and D.

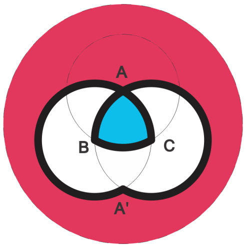

Fig. 13. G_A applied to #1, if sgn(ABC)>0 .

Proof: G_A reflect or fix each vertex and this process commutes with D and elements in **G**.
The last statement is implied by The GST Identity Theorem and

$F_a \circ G_A$ or $G_A \circ F_a : (a, b, c, A, B, C) \to (2\pi - a, \pi + b, \pi + c, 2\pi - A, \pi + B, \pi + C) \pmod{2\pi}$,

$F_b \circ G_A$ or $G_A \circ F_b : (a, b, c, A, B, C) \to (a, \pi - b, \pi + c, \pi - A, B, \pi + C) \pmod{2\pi}$,

$R_o \circ G_A$ or $G_A \circ R_o : (a, b, c, A, B, C) \to (a, \pi + b, \pi + c, A, 2\pi - B, 2\pi - C) \pmod{2\pi}$, and

since D and G_A now commutes with **G**, it is sufficient to check with type #1 triangle,

$D \circ G_A$ or $G_A \circ D : (a, b, c, A, B, C) \to (\pi - a, 2\pi - b, 2\pi - c, \pi + A, \pi - B, \pi - C) \pmod{2\pi}$.

References.

[1] Murray, D. A.: Spherical Trigonometry, Longmans, Green and Co (1908), p. 62 - 65 at
http://www.wilbourhall.org/pdfs/Spherical_Trigonometry2.pdf