Geometric Algebra formula collection v.2.0

Chapter 1 Definition of a Clifford algebra

Definition 1.1. A Clifford algebra U over B, is an algebra containing V, such that
1. $\forall_{x \in V} : x^2 = B(x, x)\, 1_U$
2. *V generates U*
3. $V \bigcap \mathbb{K}\, 1_U = \{0\}$.

Let $\mathcal{A}(V, B)$ *be the category of Clifford algebras over B.*
U is called (initial-) ***universal*** *in* $\mathcal{A}(V, B)$, *if*
4. *Any linear mapping* $f : V \to A$ *into an algebra A, such that* $f(x)^2 = B(x, x)\, 1_A$, *has a unique exacortzen@gmail.-comfacortzen@gmail.comtension to algebra morphism* $F : U \to A$. *This extension is called the universal extension.*

Theorem 1.2. Assume algebras U_i *are universal in* $\mathcal{A}(V_i, B_i)$, $f : V_1 \to V_2$ *is* $\mathbb{K}$*-linear and* $f : V_1 \to V_2$ *is* $\mathbb{K}$*-linear. Then f has a unique extension,* $F : U_1 \to U_2$ *to an algebra morphism, which is an isomorphism, if f is bijective.*

Corollary 1.3. An universal algebra in $\mathcal{A}(V, B)$ *is uniquely determined aside from isomorphisms fixing V.*

Corollary 1.4. Assume algebras U_i *are universal in* $\mathcal{A}(V_i, B_i)$ *and* $f_i : V_i \to V_{i+1}$ *is* $\mathbb{K}$*-linear.*
Let $F_i : U_i \to U_{i+1}$ *be the unique extensions to algebra morphisms.*
Then the unique extension of $f_k \circ \ldots \circ f_1$ *to an algebra morphism is* $F_k \circ \ldots \circ F_1$.

Chapter 2 Construction of a universal Clifford algebra

Definition 2.1. Let for sets $H, J \in \mathcal{F}$

$$\alpha(H, J) = \Pi\,(-1) \text{ for } (h, j) \in H \times J \text{ and } h > j$$
$$\beta(H, J) = \Pi\, q(e_i) \text{ for } i \in H \bigcap J\ ,$$
$$\sigma = \alpha\, \beta$$

Theorem 2.2. Define a product $(X, Y) \to X\,Y$ *in W by* $e_H\, e_J = \sigma(H, J)\, e_{H \triangle J}$ *and bilinearity. Then W becomes a Clifford algebra in* $\mathcal{A}(V, B)$.

Lemma 2.3. $\sigma(H, J)\, \sigma(H \triangle J, K) = \sigma(H, J \triangle K)\, \sigma(J, K)$,

Corollary 2.4. $e_I = \Pi_I\, e_i$.

Theorem 2.5. 1. W is universal in $\mathcal{A}(V, B)$.
Hence W is uniquely determined by universality in $\mathcal{A}(V, B)$ *aside from isomorphism.*
W is denoted $C\ell(B, V)$ *or* $C\ell(B)$.
2. If $(a_i \mid i \in M')$ *be an orthogonal basis for V, then* $(a_K \mid K \subseteq M',\ K \text{ finite})$ *is a basis for* $C\ell(B, V)$.
3. The Clifford product is independent of selection of orthogonal basis, if the product is constructed in $C\ell(B, V)$.

Definition 2.6. Let $\mathbb{R}^{p,q,r}$ *be a real vector space of dimension* $n = p + q + r$ *with a symmetric bilinearform B that in diagonal form has* (p, q, r) *times* $(1, -1, 0)'$*s respectively in that order. To this correspond a Clifford algebra* $\mathbb{R}_{p,q,r}$. *If* $r = 0$, *r can be omitted.*

The complex case $\mathbb{C}_{p,r}$ *is defined likewise, but without q and* -1.

Chapter 3 The Grassmann algebra over *V*

Theorem 3.1. In $\Lambda(V)$ *define the submodule of elements of grade* $r \in \mathbb{Z}$ *by*
$\Lambda_r(V) = \text{span}\,\{\wedge_{i=1}^r a_i \mid a_i \in V\}$ *for* $r \geq 0$ *and otherwise* $\Lambda_r(V) = \{0\}$.
This makes $\Lambda(V)$ *a graded algebra, as obviously* $\Lambda_r(V) \wedge \Lambda_s(V) \subseteq \Lambda_{r+s}(V)$.
Also define $x \to \langle x \rangle_r$, *as the projection on* $\Lambda_r(V)$ *along* $\oplus_{i \neq r} \Lambda_i(V)$, *and* $\langle x \rangle = \langle x \rangle_0$.
Set $\Lambda_{<p}(V) = \oplus_{i<p} \Lambda_i(V)$, *and also* $x \to \langle x \rangle_R = \oplus_{r \in R} \langle x \rangle_r$, *where R is a subset of* $\mathbb{Z}$.
Then

1. $\Lambda(V) = \oplus_r \Lambda_r(V)$ *and* $\Lambda_r(V) \wedge \Lambda_s(V) = \Lambda_{r+s}(V)$.
   If $|M|$ *is finite, then* $\text{rank}(\Lambda_{|M|}(V)) = 1$ *and* $\Lambda_r(V) = 0$ *for* $r > |M|$
2. $x \wedge x = 0$ *and* $x_1 \wedge x_2 = -x_2 \wedge x_1$
3. $x_1 \wedge x_2 \wedge \ldots \wedge x_p$ *is multilinear and alternating in the x-variables*

Theorem 3.2. (Extension by outermorphism). A $\mathbb{K}$*-linear mapping* $f : V_1 \to V_2$ *has a unique extension,*$f_\wedge : \Lambda(V_1) \to \Lambda(V_2)$ *to an algebra morphism, which is grade preserving. Moreover* $f_\wedge$ *is bijective, iff f is.*
Proof: As $f(x) \wedge f(x) = 0$, the assertion follows from universal extension, which is grade preserving, as an algebra morphism.

Theorem 3.3 (The Invariant basis *property). Two bases for V have the same finite size or are both infinite.*

Theorem 3.4. Assume M is finite. Then with respect to any basis

1. $|M| = \text{rank}(V)$ *and* $|M| = \max\,\{r \mid \Lambda_r(V) \neq 0\}$
2. $\Lambda_{|M|}(V) = \mathbb{K}\, e_M$
3. $\text{rank}(\Lambda_r(V)) = \binom{|M|}{r}$
4. $\text{rank}(\Lambda(V)) = 2^{|M|}$ *and* $\text{rank}(\Lambda(V)^+) = \text{rank}(\Lambda(V)^-) = 2^{|M|-1}$

Theorem 3.5. From $\mathcal{C}\ell(B, V)$ *any non isomorphic Clifford algebra A in* $\mathcal{A}(V, B)$ *can be found as a quotient* $\mathcal{C}\ell(B, V)/\mathcal{I}$ *with ideal* $\mathcal{I} \neq \{0\}$.
If M is finite, then *A in* $\mathcal{A}(V, B)$ *is non-universal* $\Leftrightarrow \text{rank}(A) = 2^k$ and $k < |M|$.

Definition 3.6. The geometric algebra $\mathcal{G}(B, V)$ *or* $\mathcal{G}(V)$ *is the double algebra of* $\mathcal{C}\ell(B, V)$ *and* $\Lambda(V)$ *in the same space.*
For $x_i \in V$ *set* $x_I = \Pi_{i \in I, \uparrow}\, x_i$ *and* $x_{\wedge I} = \wedge_{i \in I, \uparrow} x_i$. *By construction* $e_{\wedge I} = e_I$.

Chapter 4 Morphisms

Definition 4.1. To every algebra A is in the same linear space associated an opposite algebra A^o *with multiplication* $X \circ Y = YX$.
The linear identity $A \to A^o$ *is an anti-automorphism, and is also denoted* $o : A \to A^o$. *Moreover* $A^{oo} = A$ *and* $o^2 = \text{id}_A$.
That this multiplication makes A^o an algebra is easily verified, and also that $A^{oo} = A$ and $o^2 = \text{id}_A$.

Theorem 4.2. For any algebra A over $\mathbb{K}$ *and any linear mapping* $f : V \to A$ *such that* $f(x)^2 = B(x, x)\, 1_A$, *there* *exists a unique algebra anti-morphism* $F^o : C\ell(B, V) \to A$*, which extends f. This extension is also called the universal anti-extension.*

Corollary 4.3. Assume algebras U_i *belongs to* $\mathcal{A}(V_i, B_i)$ *and* $f_i : V_i \to V_{i+1}$ *is* $\mathbb{K}$*-linear.*
Then f_i *has a unique extension,* $U_i \to U_{i+1}$ *to an algebra anti-morphism, which is an anti-isomorphism, if* f_i *is bijective.*
Let $F_i : U_i \to U_{i+1}$ *be a morphism or an anti-morphism and* $F = F_k \circ \ldots \circ F_1$*. If the number of anti-morphism in the composition is odd, then F is anti-morphism, and otherwise a morphism.*

Definition 4.4. Let U be a Clifford algebra in $\mathcal{A}(V, B)$*, not necessarily universal.*
As proved a linear mapping $f : V \to V$ *has at most one extension to an automorphism or anti-automorphism of U.*
If they exists,
the main or grade automorphism $X \to \hat{X}$ *is the extension of* $f : V \to V$*,* $f(x) = -x$ *to an automorphism of U.*
the reversion $X \to \tilde{X}$ *is the extension of* $f : V \to V$*,* $f(x) = x$ *to an anti-automorphism of U.*
the Clifford conjugation $X \to \overline{X}$ *is the extension of* $f : V \to V$*,* $f(x) = -x$ *to an anti-automorphism of U.*

Theorem 4.5. $C\ell_V(B)$ *is extended to a geometric algebra to make the grade concept available.*
1. In $C\ell_V(B)$ *the main automorphism, the reversion, and the Clifford conjugation exists.*
2. For the main automorphism holds $\text{grade}(X) = r \Rightarrow \hat{X} = (-1)^r X$
3. For the reversion holds $(X\,Y)^\sim = \tilde{Y}\,\tilde{X}$, $(a_1\, a_2 \ldots a_r)^\sim = a_r \ldots a_2\, a_1$ *for* $a_i \in V$ *and* $\text{grade}(X) = r \Rightarrow \tilde{X} = (-1)^{r(r-1)/2} X$
4. For the Clifford conjugation holds $\overline{X} = \hat{X}^\sim$ *and* $\text{grade}(X) = r \Rightarrow \overline{X} = (-1)^{r(r+1)/2} X$
5. These three mappings are grade preserving, involutions, commuting and independent of the B. Each one is the composition of the two others.

Chapter 5 Basic structure of Geometric algebra

Definition 5.1. Set $\chi_S = 1$*, if the proposition S is true, and else zero. Define in* $\mathcal{G}(B, V)$ *compositions* $\cdot, \rfloor$ *and* $\lfloor$ *by bilinearity by*
$e_H \cdot e_J = \chi_{H=J}\; e_H\, e_J$*, the scalar product,*
$e_H \rfloor e_J = \chi_{H \subseteq J}\; e_H\, e_J$*, the left contraction,*
$e_H \lfloor e_J = \chi_{H \supseteq J}\; e_H\, e_J$*, the right contraction.*
We already know from the product definition based on the α and β functions that
$e_H \wedge e_J = \chi_{H \cap J = \emptyset}\; e_H\, e_J$

Theorem 5.2. In a geometric algebra $\mathcal{G}(B, V)$ *holds*

1. $x\,X = x \rfloor X + x \wedge X$ *and* $X\,x = X \wedge x + X \lfloor x$
   $x \cdot y = B(x, y)\, 1_{\mathcal{G}}$,
2. *If* $\text{grade}(X) = r$ *and* $\text{grade}(Y) = s$*, then*
   $X \cdot Y = \langle X\,Y \rangle = \langle Y\,X \rangle = Y \cdot X$, $X \rfloor Y = \langle X\,Y \rangle_{s-r}$, $X \lfloor Y = \langle X\,Y \rangle_{r-s}$ *and* $X \wedge Y = \langle X\,Y \rangle_{r+s}$
   The three main involutions are symmetric, $\hat{X} \cdot Y = X \cdot \hat{Y}$, $\tilde{X} \cdot Y = X \cdot \tilde{Y}$, $\overline{X} \cdot Y = X \cdot \overline{Y}$
3. *If* $\text{grade}(X) = r$ *and* $\text{grade}(Y) = s$*, then*
   $r \neq s \Rightarrow X \cdot Y = 0$, $Y \lfloor X = \left(\tilde{X} \rfloor \tilde{Y}\right)^\sim = (-1)^{(s+1)\,r}\, X \rfloor Y$ *and* $X\,Y = \Sigma_{i=|r-s| \text{ step } 2}^{r+s} \langle X\,Y \rangle_i$
4. $(X \wedge Y) \rfloor Z = X \rfloor (Y \rfloor Z)$ *and* $(X \wedge Y) \cdot Z = X \cdot (Y \rfloor Z)$
5. $x \rfloor (X\,Y) = (x \rfloor X)\, Y + \hat{X}\, (x \rfloor Y)$

$x \wedge (X\,Y) = (x \rfloor X)\,Y + \hat{X}\,(x \wedge Y)$
$x \wedge (X\,Y) = (x \wedge X)\,Y - \hat{X}\,(x \rfloor Y)$
$x \rfloor (X\,Y) = (x \wedge X)\,Y - \hat{X}\,(x \wedge Y)$

6. $x \rfloor (X \wedge Y) = (x \rfloor X) \wedge Y + \hat{X} \wedge (x \rfloor Y)$
$x \wedge (X \rfloor Y) = (x \rfloor X) \rfloor Y + \hat{X} \rfloor (x \wedge Y)$

7. $x \rfloor (x_1\, x_2 \ldots x_p) = \sum_{k=1}^{p} (-1)^{k-1}\, x_1\, x_2 \ldots (x \rfloor x_k) \ldots x_p$

8. $x \rfloor (x_1 \wedge x_2 \wedge \ldots \wedge x_p) = \sum_{k=1}^{p} (-1)^{k-1}\, x_1 \wedge x_2 \ldots \wedge (x \rfloor x_k) \ldots \wedge x_p$

9. $x_1, x_2, \ldots, x_p$ *are pairwise orthogonal* $\Rightarrow \Pi_{i=1}^{p}\, x_i = \wedge_{i=1}^{p}\, x_i$

10. $x\,X - \hat{X}\,x = 2\,x \rfloor X$ *and* $x\,X + \hat{X}\,x = 2\,x \wedge X$

11. $\forall_{x \in V}\,(x \wedge A = 0) \Leftrightarrow (A \in \mathbb{K}\,e_M$, *if* $|M|$ *is finite, and otherwise* $A = 0)$.
Assume $\mathbb{K}$ *is a field or the weaker condition* $\mu\, e_i^2 = 0 \Rightarrow (\mu = 0 \text{ or } e_i^2 = 0)$ for $i \in M,\ \mu \in \mathbb{K}$. *Then*
$\forall_{x \in V}\,(x \rfloor A = 0) \Leftrightarrow A \in \mathcal{G}(V_0)$, *where* V_0 *is the radical or kernel of B.* (*NB: Always* $\mathbb{K} \subseteq \mathcal{G}(V_0)$)

12. $(x_1 \wedge x_2 \ldots \wedge x_r) \cdot (y_r \wedge \ldots \wedge y_2 \wedge y_1) = \Sigma_\sigma\, s_\sigma\, (x_1 \cdot y_{\sigma(1)}) \ldots (x_r \cdot y_{\sigma(r)})$
where summation is over all permutations σ *of* $\{1, \ldots, r\}$.

13. *Factor expansion of* $x_{\wedge K}$: *Let* $X = x_{\wedge K}$ *and* $B \in \Lambda_s(V)$. *If* $\tau_H = \alpha(H, K \backslash H)\,(B \cdot x_{\wedge H}) \in \mathbb{K}$ *), *then*
$$B \rfloor X = \Sigma_{H \subseteq K,\, |H| = s}\ \tau_H\, x_{\wedge K \backslash H}$$

*) α *is from definition 2.1:* $\alpha(H, J) = \Pi\,(-1)$ for $(h, j) \in H \times J$ and $h > j$

Determinant Theorem 5.3. *Let* $f: V \to V$ *be linear mapping*.
1. *If V has a finite basis, then the determinant* $\det(f)$ *is defined by* $F(e_M) = \det(f)\, e_M$ *independent of basis.*
2. *Moreover* $(f \circ g)_\wedge = f_\wedge \circ g_\wedge$, $\det(f \circ g) = \det(g)\det(f)$, *and* $\det(f^{-1})\det(f) = 1$ *when f is bijective.*
3. *Assume* $m = |M|$, $M = \{1, \ldots, m\}$ *and* $f(a_s) = \Sigma_i\, \theta_s^i\, a_i$ *in some basis* $(a_i \mid i \in M)$. *Then*
$\det(f) = \Sigma_\sigma\, \mathrm{sign}(\sigma)\,(\theta_{\sigma(1)}^{1} \ldots \theta_{\sigma(m)}^{m})$, *where summation is over all permutations* σ *of M.*

Automorphism Theorem 5.4. *Let* $f: V \to V$ *be linear mapping, such that* $f(x)^2 = B(x, x)\, 1_A$. *Then f has two universal extensions:*
To an outermorphism $f_\wedge : \Lambda(V) \to \Lambda(V)$, *and to Clifford algebra isomorphism* $F: C\ell_V(B) \to C\ell_V(B)$.
Assume $B(f(x), f(y)) = B(x, y)$, *or* $\mathbb{K}$ *is a field not of characteristic 2. Then* $f_\wedge$ *is called the universal extension of f to* $\mathcal{G}(B, V)$, *as*
$F = f_\wedge$, *and thus is an outermorphism, and furthermore grade preserving, an orthogonal isomorphy, an isomorphy for* $\rfloor$ *and* $\lfloor$, *and commutes with the three main involutions.*

Anti-automorphism Theorem 5.5. *Let* $f: V \to V$ *be linear mapping, such that* $f(x)^2 = B(x, x)\, 1_A$. *Then f has two universal anti-extensions:*
To an anti-outermorphism $f_\wedge^\tau : \Lambda(V) \to \Lambda(V)$, *and to Clifford algebra anti-isomorphism* $F^\tau : C\ell_V(B) \to C\ell_V(B)$.
Assume $B(f(x), f(y)) = B(x, y)$, *or* $\mathbb{K}$ *is a field not of characteristic 2. Then* $f_\wedge^\tau$ *is called the universal anti-extension of f to* $\mathcal{G}(B, V)$, *as*
1. $F^\tau = f_\wedge^\tau$, *and thus is an anti-outermorphism, grade preserving, an orthogonal isomorphism and commutes with the three main involutions. Furthermore* $F^\tau(X \rfloor Y) = F^\tau(Y) \lfloor F^\tau(X)$ *and* $F^\tau(Y \lfloor X) = F^\tau(X) \rfloor F^\tau(Y)$.
2. *If V has a finite basis, then* $F^\tau(e_M) = (-1)^{|M|\,(|M|-1)/2}\det(f)\, e_M$.

Theorem 5.6. Assume B is regular and $\mathbb{K}$ *is a field of characteristic* $\neq 2$. *Then*
A is universal in $\mathcal{A}(V, B) \Leftrightarrow$ *A has a main automorphism*

Definition 5.7. *A list of elements in a module is linear independent, if the only (finite) linear combination of the elements giving zero is that with zero factors. Linear dependent means not linear independent.*
Obviously holds: A list of elements is linear independent ⇔ *every finite sublist is linear independent*

Theorem 5.8. *Let H be finite. Then*
A: $(x_h \mid h \in H)$ *is linear independent* ⇔ B: $x_{\wedge H}$ *is linear independent* ⇔ C: $(x_{\wedge K} \mid K \subseteq H)$ *is linear independent*

Corollary 5.9. $S = (x_1, x_2 \ldots, x_p)$ *is linear independent* ⇔ $S_\wedge = x_1 \wedge x_2 \wedge \ldots \wedge x_p$ *is linear independent*

Corollary 5.10. *Allow* $H_0 \subseteq M$ *to be infinite. Then*
$(x_h \mid h \in H_0)$ *is linear independent* ⇔ $(x_{\wedge K} \mid K \subseteq H_0,\ K \text{ finite})$ *is linear independent*

Definition 5.10. *If U is a submodule of V, then set* $\Lambda(U) = \operatorname{span}\{\wedge_{i=0}^{m} U \mid m \in \mathbb{N}\}$, *which obviously is the Grassmann-algebra generated by U.*
Also set $\Lambda_{>0}(U) = \operatorname{span}\{\wedge_{i=1}^{m} U \mid m \in \mathbb{N}\}$.

Theorem 5.11. *Let* $A = a_{\wedge H}$ *be a blade.*
Define modules $V_A = \{x \in V \mid x \wedge A = 0\}$, $V_{A\perp} = \{x \in V \mid \forall_{h \in H}\ x \cdot a_h = 0\}$ *and set* $A^{\shortparallel} = \Lambda(V_A)$, $A^{\perp} = \Lambda_{>0}(V_{A\perp})$.
Obviously $\operatorname{span}\{a_h \mid h \in H\} \subseteq V_A$ *implying* $\Lambda(\{\operatorname{span}\{a_h \mid h \in H\}) \subseteq A^{\shortparallel}$. *Moreover also* $V_1 = \{0\}$ *and* $V_{1\perp} = V$ *and* $1^{\shortparallel} = \{0\}$, $1^{\perp} = \Lambda_{>0}(V)$.
Omitting $\shortparallel$ *like in* $X \subseteq A$ *instead of* $X \subseteq A^{\shortparallel}$ *can be used, if it is clear that A means an algebra and not a blade.*
Inclusions like $X \subseteq A^{\shortparallel}$ *or* $X \subseteq A^{\perp}$ *may be used for elements, as in* $e_1 \subseteq A^{\shortparallel}$ *meaning* $\{e_1\} \subseteq A^{\shortparallel}$
Then
1. $X \in \operatorname{span}\{a_{\wedge H_1}, \ldots, a_{\wedge H_k} \mid \forall_{h \in H}\ H_h \subseteq H\} \Rightarrow X \subseteq A^{\shortparallel}$
   NB: The opposite inclusion is true, if V is a vectorspace; but not generally for modules.
2. $B \rfloor A \subseteq A^{\shortparallel}$
3. $C \rfloor A = C\,A$, *when* $C \subseteq A^{\shortparallel}$
4. $A^2 = A \rfloor A = A \cdot A$ *and* A *is invertible* ⇔ $A \cdot A$ *invertible* ⇒ $A^{-1} = A/(A \cdot A)$
5. *Assume A is invertible. Then* $\operatorname{span}\{a_h \mid h \in H\} = V_A$.

NB: In Corollary 6.2.5 is proved: A invertible ⇒ $V = V_A \oplus V_{A\perp}$

Lemma 5.12. *Let A be a blade. Then*
1. *If* $C \subseteq A^{\shortparallel}$:
   $(C \rfloor B)\,A = C \wedge (B\,A)$
   $(C \rfloor B) \rfloor A = C \wedge (B \rfloor A)$
   $(C \wedge B)\,A = C \rfloor (B\,A)$
   $(C\,B) \rfloor A = C\,(B \rfloor A)$
2. *If* $C \subseteq \mathbb{K} + A^{\perp}$:
   $(C \rfloor B)\,A = C \rfloor (B\,A)$
   $(C \rfloor B) \wedge A = C \rfloor (B \wedge A)$
   $(C \wedge B)\,A = C \wedge (B\,A)$
   $(C\,B) \wedge A = C\,(B \wedge A)$

Chapter 6 Geometric transformations

Theorem 6.1. For a h-blade A assume $\rho = A \cdot A$ *is invertible, thus* $A^{-1} = \rho^{-1} A$.
Define the projection on A as $P_A(X) = P(X) = (X \rfloor A) \rfloor A^{-1}$.
1. Then P is grade preserving,
2. $P(X) = (X \rfloor A)\, A^{-1}$, $\; P(X) = \rho^{-1} A\, (A \lfloor X) = \rho^{-1} A \lfloor (A \lfloor X)$
3. $X \subseteq A \Rightarrow P(X) = X$, $P(X) \subseteq A$, $P^2(X) = P(X)$
4. $P(\Lambda(V)) = A^{\shortparallel}$, $P(V) = V_A$, $P(A^{\perp}) = \{0\}$.
5. P is symmetric, $X \cdot P(Y) = P(X) \cdot Y$
6. Moreover P is an outermorphism.

Corollary 6.2. Define the rejection of X by A as $Q_A(X) = Q(X) = X - P_A(X)$. *Then*

Corollary 6.3. Define the projection ***along*** *A,* $\mathcal{P}^A$, *as the extension of* $Q_A(x)$ *by outermorphism. Then* $\mathcal{P}^A$ *is grade preserving, and*
1. $\mathcal{P}^A(X) = \mathcal{P}(X) = A^{-1} \rfloor (A \wedge X) = A^{-1}(A \wedge X) = (X \wedge A)\, A^{-1} = (X \wedge A) \lfloor A^{-1} \subseteq A^{\perp}$
2. $X \subseteq A^{\perp} \Rightarrow \mathcal{P}(X) = X$, $\mathcal{P}(X) \subseteq A^{\perp}$, $\mathcal{P}^2(X) = \mathcal{P}(X)$, $\mathcal{P}(A^{\shortparallel}) = \{0\}$, *and* $P_A \circ \mathcal{P}^A = \mathcal{P}^A \circ P_A = 0$
3. S*ymmetry* $X \cdot \mathcal{P}(Z) = \mathcal{P}(X) \cdot Z$

Theorem 6.4 Projection P_A^B *on A along B (with* $(A \wedge B)^{\perp}$ *fixed). Assume* $\mathbb{K}$ *is a field not of characteristic 2.*
For a r-blade A and a s-blade B let $C = A \wedge B$ *and assume C is invertible and set* $\eta = (B \wedge A) \cdot C$. *Then* $C^{-1} = (-1)^{r\,s}\, \eta^{-1}\, C$ *and*
1. $V = V_A \oplus V_B \oplus V_{\perp C}$ *as direct sum of vectorspaces, and this defines projections.* P_A^B *is the projection on* $V_A \oplus V_{\perp C}$ *along* V_B.
2. P_A^B *extended by outermorphism gives* $P_A^B(X) = \eta^{-1}(A \rfloor C)\big\rfloor (B \wedge X)$
3. $X \subseteq \Lambda(V_A + V_{\perp C}) \Rightarrow P(X) = X$, $X \subseteq B \Rightarrow P(X) = 0$, $P(X) \subseteq \Lambda(V_A + V_{\perp C})$, $P^2(X) = P(X)$, $P_A^B \circ P_B^A = P_B^A \circ P_A^B = \mathcal{P}^C$, $P_B^A + P_A^B - \mathcal{P}^C = \mathrm{id}_{\Lambda(V)}$

Theorem 6.5. For a h-blade A assume $\rho = A \cdot A$ *is invertible, such that* $A^{-1} = \rho^{-1} A$.
Define the reflection ***along*** *A by linearity and*
$$\mathcal{R}^A(X) = \mathcal{R}(X) = (-1)^{h\,r}\, A\, X\, A^{-1}, \text{ when } \mathrm{grade}(X) = r. \text{ Then}$$
1. $\mathcal{R}^A(x) = \hat{A}\, x\, A^{-1}$
$\mathcal{R}(x) = x - 2\, P(x)$
$\mathcal{R}(P(x)) = -P(x)$ *and* $\mathcal{R}(Q(x)) = Q(x)$ *justify the words "along A".*
2. Also $\mathcal{R}^2(X) = X$, *and* $\mathcal{R}(X) \cdot \mathcal{R}(Y) = X \cdot Y$.
3. Moreover $\mathcal{R}^A$ *is a Clifford algebra isomorphism and universal extension of its restriction to V.*
$\mathcal{R}$ *is e.g. grade preserving and an outermorphism, as the Automorphism Theorem 5.4 apply to* $\mathcal{R}$.
4. Also $\mathcal{R}(P(X)) = P(\hat{X})$, $\mathcal{R}\big(\mathcal{P}^A(X)\big) = \mathcal{P}^A(X)$, $\mathcal{R}$ *is symmetric,* $\mathcal{R}(X) \cdot Y = X \cdot \mathcal{R}(Y)$
5. If V has a finite basis, then $\det\big(\mathcal{R}^A\big) = (-1)^h$

Corollary 6.6. Define the reflection ***in*** *A as* $R_A(X) = R(X) = \mathcal{R}^A(\hat{X})$. *Then*
1. $R_A(X) = (-1)^{h\,r}\, A\, \hat{X}\, A^{-1}$ *and, if* $x \in V$, *then*
$R_A(x) = -\mathcal{R}^A(x) = -\hat{A}\, x\, A^{-1}$
$R(x) = -x + 2\, P(x)$
$R(P(x)) = P(x)$ *and* $R(Q(x)) = -Q(x)$ *supports the words "in A".*

2. Also $R^2(X) = X$, *and* $R(X) \cdot R(Y) = X \cdot Y$.
3. Moreover R_A *is a Clifford algebra isomorphism and universal extension of its restriction to V.*
R is e.g. grade preserving and an outermorphism, as the Automorphism Theorem 5.4 apply to R.
4. $R(P(X)) = P(X)$, $\mathcal{R}(\mathcal{P}^A(X)) = \mathcal{P}^A(\hat{X})$, $R^2(X) = X$, *R is symmetric,* $R(X) \cdot Y = X \cdot R(Y)$
5. If V has a finite basis, then $\det(R_A) = (-1)^{h+|M|}$

Definition 6.7 A versor of order h or a h-versor, $U = u_1 \dots u_h$, *as a product of invertible elements* $u_i \in V$.
1. As $U^{-1} = u_h^{-1} \dots u_1^{-1}$ *a versor transformation of* $C\ell(B, V)$ *is defined by linearity and*
$S(X) = S_U(X) = (-1)^{hr}\, U X U^{-1}$ *,when* $\text{grade}(X) = r$, *e.g.* $S(x) = \hat{U}\, x\, U^{-1}$
1. As $S_U = \mathcal{R}^{u_1} \circ \dots \circ \mathcal{R}^{u_h}$ *the Automorphism Theorem apply to* S_U, *and, if V has a finite basis, then* $\det(S_U) = (-1)^h$
2. The Clifford group Γ *is the multiplicative group of versors.*
Define Γ^+, *the rotation versors, as the versors of even order, which obviously is a subgroup of* Γ *of index 2.*
The orthogonal isomorphisms $V \to V$ *is a group under composition* $\circ$, *the orthogonal group O(B).*
The mapping $\Psi : U \to S_U$ *is a multiplicative morphism from* Γ *into the orthogonal group O(B).*

Lemma 6.8. The mapping $\Psi : U \to S_U$ *is a multiplicative morphism from* Γ *into the orthogonal group O(B).*

Theorem 6.9. Assume $\mathbb{K}$ *is a field not of characteristic 2, V has a finite basis and B is regular.*
1. Then from the Cartan-Dieudonne theorem follows that any orthogonal isomorphism of V can be expressed as $S(x) = \hat{U}\, x\, U^{-1}$, *where U is a h-versor with* $h \le n = \dim(V)$.
2. $\forall_{x \in V}\ \hat{U}\, x\, U^{-1} = \hat{T}\, x\, T^{-1}$ *imply* $T \in \mathbb{K}^\times U$.
3. $\Psi : \Gamma \to O(B)$ *is onto O(B) with kernel* $\Psi^{-1}(\text{id}_V) = \mathbb{K}^\times$
Also Ψ *maps* Γ^+ *onto* $O^+(B)$, *the orthogonal isomorphisms with determinant* 1 *called rotations.*
4. Moreover, if $\forall_{x \in V}\ \psi(x) = \hat{T}\, x\, T^{-1} \in V$, *then T is a versor.*

Corollary 6.10. The mapping $\Phi : U \to U \tilde{U} \in \mathbb{K}^\times$ *is a multiplicative morphism,* $\Phi(\Gamma)$ *is a multiplicative group, and* $\Phi(\Gamma) = \Phi(\Gamma)\, \mathbb{K}^{\times 2}$.
2. Assume $\mathbb{K}^\times = S \times (\mathbb{K}^\times)^2$ *as direct product of multiplicative subgroups S and* $(\mathbb{K}^\times)^2$, *like e.g.* $\mathbb{R}^\times = \{\pm 1\} \times (\mathbb{R}^\times)^2$ *or* $\mathbb{C}^\times = \{1\} \times (\mathbb{C}^\times)^2$.
Then each $U \in \Gamma$ *can be normalized as t U, such that* $\Phi(t\, U) \in S$, *and t is unique apart from a factor* ± 1.
Define $\text{pin}(B) = \Phi^{-1}(S)$ *and* $\text{spin}(B) = \text{pin}(B) \cap \Gamma^+$, $\text{pin}^+(B) = \Phi^{-1}(1)$ *and* $\text{spin}^+(B) = \text{pin}^+(B) \cap \Gamma^+$.
If $U \in \text{pin}(B)$, *then* $S_U(x) = s^{-1}\, \hat{U}\, x\, \tilde{U}$, *where* $s = \Phi(U) \in S$

Chapter 7 Finer structures in Clifford algebra

Theorem 7.1. Let (a_i) *be an orthogonal basis for V ,* $V_\kappa = V \oplus \mathbb{K}\, a_\kappa$, a_κ *orthogonal to V,* $a_\kappa^2 = \varepsilon$ *invertible, and* B_κ *the extension of B to* V_κ.
Define a linear mapping $f : V \to C\ell(B_\kappa)^+$ *by* $u \to u\, a_\kappa$. *Then f extends uniquely to an algebra isomorphism* $F : C\ell(-\varepsilon B) \to C\ell(B_\kappa)^+$.

Definition 7.2. An algebra A is called simple, if A has no twosided ideals other than 0 and A.
The center $Z = Z(A)$ *of an algebra A consists of the elements commuting with all the elements of A.*

Theorem 7.3. Assume $\mathbb{K}$ *is a field of characteristic* $\neq 2$. *Then,*
if $|M|$ *is finite and odd, then* $Z = Z(C\ell(B)) = \mathbb{K}\, e_M + \mathcal{G}(V_0)^+$, *and otherwise* $Z = \mathcal{G}(V_0)^+$, where V_0 *is the radical or*

kernel of B.

Lemma 7.4. For algebra A and $f \in A$ assume $f^2 = f$. Then
1. $(1-f)^2 = (1-f)$ *and* $f(1-f) = 0$.
2. $\mathcal{I}_- = A f$ *is a left ideal.*
3. $P_- : X \to X f$ *is an algebraic projection onto* $\mathcal{I}_-$*, such that* $P_-^2 = P_-$, $(1-P_-)^2 = (1-P_-)$ *and* $P_-(1-P_-) = 0$ with $1 = \mathrm{id}_V$.
Moreover $X \in \mathcal{I}_- \Rightarrow P_-(X) = X$
4. If $f \in Z(A)$*, then* $\mathcal{I}_-$ *is as a twosided ideal, and* $P_- : A \to \mathcal{I}_-$ *is an algebra homomorphism.*
As $(1-f)$ *has the same properties as those mentioned of f, it give likewise rise to objects* $\mathcal{I}_+ = 1 - \mathcal{I}_+$ *and* $P_+ = 1 - P_-$.
Analogous statements to (1-3) holds and furthermore
5. $\mathcal{I}_- \oplus \mathcal{I}_+ = A$

Theorem 7.5. Assume $\mathbb{K}$ *is a field of characteristic≠2 and B is regular. Then*
1. If $|M|$ *is even or infinite, then* $C\ell_V(B)$ *is simple.*
2. Assume $|M|$ *finite and odd. Then*
$\mathcal{I}$ *is a non-trivial ideal* $\Leftrightarrow \exists_\lambda : \mathcal{I} = C\ell(B)(1 + \lambda e_M)$ *and* $\lambda^2 e_M^2 = 1$.
3. Assume $|M|$ *finite and odd, and* $\lambda^2 e_M^2 = 1$. *Then*
(3a) $f_\pm = (1 \pm \lambda e_M)/2 \in Z(C\ell(B))$, $f_\pm f_\mp = 0$ *and* $f_\pm^2 = f_\pm$.
This gives projections and algebra homomorphisms $P_\pm(X) = X f_\pm$ *onto proper ideals* $\mathcal{I}_\pm = P_\pm(C\ell(B))$*, such that* $P_- + P_+ = \mathrm{Id}_{C\ell(B)}$, $P_- P_+ = 0$, $P_\pm^2 = P_\pm$, *and* $\mathcal{I}_- \oplus \mathcal{I}_+ = C\ell(B)$.
(3b) $P_\pm(X) \cdot Y = X \cdot P_\pm(Y)$
(3d) $C\ell(B)^+$ *isomorphic to each ideals* $\mathcal{I}_\pm$ *by the restriction of* $P_\pm$ *to* $C\ell(B)^+$.
(3e) $C\ell(B)^+$ *and* $\mathcal{I}_\pm$ *are all simple.*
(3f) The only non-trivial ideal in $C\ell_V(B)$ *are* $\mathcal{I}_-$ *and* $\mathcal{I}_+$.

Theorem 7.6. Assume B is regular and $\mathbb{K}$ *is a field of characteristic≠2, that* $|M|$ *is finite* >1 *and odd, and also that* $\lambda \in \mathbb{K}$ *can be found, such that* $\lambda^2 e_M^2 = 1$. *Then*
1. In $\mathcal{A}(V, B)$ *exists besides* $C\ell_V(B)$ *only the algebras* $U_\pm = C\ell(B)/\mathcal{I}_\pm$.
2. If $\tilde{e_M} = e_M$*, then* $C\ell(B)/\mathcal{I}_\pm$ *has reversion and no conjugation.*
Otherwise, if $\overline{e_M} = e_M$*, then* $C\ell(B)/\mathcal{I}_\pm$ *has conjugation and no reversion.*

Lemma 7.7. Let $x_i \in V$*, then* $x_1 x_2 \dots x_p - x_1 \wedge x_2 \wedge \dots \wedge x_p \in \Lambda_{<p}(V)$.

Theorem 7.8. Let $(x_i \mid i \in I)$ *be linear independent in V.*
1. Then $(x_K \mid K \subseteq I, K \text{ finite})$ *is linear independent.*
2. Assume $(x_i \mid i \in I)$ *is a basis for V, and* $K \subseteq I$*, K finite.*
Then the quantization transformation $f : \mathcal{G}(V) \to \mathcal{G}(V)$ *is well-defined by linearity and* $f(x_{\wedge K}) = x_K$.
Moreover (x_K) *is a basis for* $\mathcal{G}(V)$.

Definition 7.9. In $C\ell(B)$ *define parity of X by* $\mathrm{par}(X) = p \Leftrightarrow \mathrm{grade}(X) \equiv p \pmod 2$.
Also set $C\ell(B)^- = \{X \mid \mathrm{par}(X) = 1\}$ *and* $C\ell(B)^+ = \{X \mid \mathrm{par}(X) = 0\}$
Parity makes $C\ell(B)$ *a graded algebra:* $\mathrm{par}(X) = r$ and $\mathrm{par}(Y) = s \Rightarrow \mathrm{par}(X Y) = r + s \bmod(2)$.

Definition 7.10. To every Clifford algebra $C\ell(B, V)$ *is associated a twisted algebra* $C\ell(B, V)^{\text{tw}}$ *in the same linear space with multiplication defined by linearity and* $X \,{}^\tau\, Y = (-1)^{rs} X\,Y$, *when* $\text{par}(X) = r$ *and* $\text{par}(Y) = s$

1. This gives an algebra structure, such that $x \,{}^\tau\, x = -x^2$ *for* $x \in V$*. The twisted of the twisted algebra is the original.*

2. The universal extension of id_V *is an algebra isomorphism* $F : C\ell\,(-B, V) \to C\ell(B, V)^{\text{tw}}$.

Chapter 8 Chevalley's construction of Clifford algebras from tensor algebras

Theorem 8.1. Let $\mathcal{T} = \mathcal{T}(V, \otimes)$ *be the tensor algebra over* V*. For any algebra* A *over* $\mathbb{K}$ *and any linear mapping* $\tau : V \to A$, *there is a unique algebra morphism* $\mathrm{T} : \mathcal{T} \to A$ *that extends* τ.

Definition 8.2. Let $\mathcal{I} = \mathcal{I}(V, B)$ *be the two-sided ideal in* $\mathcal{T} = \mathcal{T}(V)$ *generated by* $\mathcal{S} = \{x \otimes x - B(x, x)\, 1_{\mathcal{T}} \mid x \in V\}$.
The Clifford algebra $CC\ell_V(B)$ *is then defined as the quotient algebra* $CC\ell = \mathcal{T} / \mathcal{I}$ *together with* $\hat{\pi} : \mathcal{T} \to CC\ell$ *the canonical algebra morphism.*

Definition 8.3. Let $\mathcal{K}(V, B)$ *be the category of linear mappings* f *from* V *into an algebra* A, *such that* $f(x)^2 = B(x, x)\, 1_A$.
A mapping $\omega : V \to U$ *in* $\mathcal{K}(V, B)$ *is said to be universal, if for every linear mapping* $f : V \to A$ *in* $\mathcal{K}(V, B)$, *there is a unique algebra morphism* $F : U \to A$ *such that* $F \circ \omega = f$. (i.e. $f : V \xrightarrow{\omega} U \xrightarrow{F} A$)

Theorem 8.4.
1. $\pi = \hat{\pi}\,|_V$ *from definition 8.2 is a universal object in* $\mathcal{K}(V, B)$.
2. Assume B *is symmetric.*
Then $\pi : V \to CC\ell$ *is injective and has an extension* $G : C\ell_V(B) \to CC\ell$ *to an algebra isomorphism, and therefore* $G(1_{C\ell_V(B)}) = 1_{CC\ell}$.
Therefore a Cliford algebra in the version presented in definition 1.1 is also a Chevalley Cliford algebra.